

Fractional Fourier Transform Based OFDMA for Doubly Dispersive Channels

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Abstract— The performance of Orthogonal Frequency Division Multiple Access (OFDMA) system degrades significantly in doubly dispersive channels. This is due to the fact that exponential sub-carriers do not match the singular functions of this type of channels. To solve this problem, we develop a system whose sub-carriers are chirp functions. This is equivalent to exploiting Fractional Fourier Transform (FrFT) instead of Fourier Transform (FT) in the structure of the aforementioned transmission scheme. We name the new system FrFT-OFDMA. The optimal angle of fractional Fourier transform for each user depends on its channel parameters. Thus, the angles of transform for different users are not necessarily identical. This destroys the orthogonality of users and generates Multi-User Interference (MUI). By analyzing MUI, we introduce quasi-orthogonality conditions where interference is negligible despite different angles of transform. For non-orthogonal users, we propose a method to mitigate MUI. We present the efficiency of this method through comparative performance evaluation of the conventional system based on FT and the new system based on FrFT. We show that our proposed transmission scheme outperforms the traditional OFDMA system significantly in doubly dispersive channels and channels impaired by frequency offset.

Index Terms— Doubly dispersive channels, Fractional Fourier transform, OFDMA.

I. INTRODUCTION

Orthogonal Frequency Division Multiple Access (OFDMA) is a type of Generalized Multi-Carrier (GMC) techniques adopted for downlink transmission in Long Term Evolution Advanced (LTE-Advanced) standard [1][2]. The orthogonality of sub-carriers in this air interface prevents Inter Carrier Interference (ICI) and provides appealing attributes such as robustness to frequency selectivity. In doubly dispersive channels, however, the orthogonality of sub-carriers is violated which generates ICI [3]-[6]. To remove ICI completely in these channels, the complexity of the equalization process significantly increases compared to the simple frequency domain equalization utilized in OFDMA system. For instance, the complexity of the equalization process based on Minimum Mean Square Error (MMSE) criterion will be in order of N^2 where N denotes the number of sub-carriers. To reduce

the complexity, in [7] a two stages equalizer has been proposed whose overall complexity is in the order of $D^2N + N$ where D is determined by the Doppler frequency. Thus, the complexity of this equalizer increases significantly with the Doppler frequency.

To overcome the problem of ICI in doubly dispersive channels, we propose a multi-user system based on Fractional Fourier Transform (FrFT). We name the proposed scheme FrFT-OFDMA. The structure of this system is similar to that of OFDMA where Fast Fourier Transform (FFT) and Inverse FFT (IFFT) are replaced by FrFT and Inverse FrFT (IFrFT). This is equivalent to exploiting sub-carriers that are chirp functions whose rate equals cotangent of the angle of FrFT. Utilizing the chirp functions in OFDM systems has been proposed in [8] where it is shown that proper selection of angle of transform significantly reduces ICI power. Several algorithms have been proposed to efficiently implement Discrete FrFT (DFrFT) to obtain a complexity comparable to that of FFT [9]-[12]. Different aspects of FrFT-OFDM systems have also been investigated, for instance, [13] has analyzed Signal to Interference Ratio (SIR) of these systems in different types of channels. This idea has also been generalized to block-wise single carrier systems as [14] has proposed a Single Carrier system with Frequency Domain Equalization (SC-FDE) based on the FrFT for aeronautical channels. Despite the substantial amount of research existing on FrFT-OFDM systems, there is very limited work on the multi-user scenario of these systems. One of the works on this subject is [15] which assigns a distinct angle of transform to each cell to reduce inter cell interference. This angle is common for all users within the cell which guarantees the orthogonality of those users. [16] considers the FrFT order as a new resource to suppress co-channel interference in multiuser SISO-BOFDM. It effectively allocates FrFT orders to different co-channel mobile stations to enable co-channel mobile stations to use the same frequency in the same time slot. However, since the optimum angle of transform for each user depends on its channel, the proposed systems in both papers lose performance due to ICI generated due to non-optimality of angles of transform in doubly dispersive channels. There are also works on Carrier Frequency Offset (CFO) estimation like [17], which uses partial FFT demodulation instead of full FFT demodulation at the receiver to estimate carrier frequency offset in uplink OFDMA. But it does not utilize FrFT with optimum angle of transform to reduce the effect of frequency offset.

In our proposed multi-user scheme, to overcome the interference due to frequency offset and double dispersivity of the channel, we optimize users angles of transform based on their corresponding channel state information. Thus, the orthogonality of users deteriorates and multi-user interference occurs. We show that MUI power depends on the difference between cotangent of angles of transform. Based on the value of this difference, we define quasi-orthogonality conditions where MUI is negligible. When users are neither orthogonal nor quasi-orthogonal, the sub-carriers assigned to a user contaminate the adjacent spectrum allocated to other users. To combat this phenomenon, we use a spectral guard interval between different users. To show the efficiency of our proposed schemes, we consider the proposed systems over a two-ray double dispersive channel and a time-invariant

channel with frequency offset. We show that our proposed scheme requires about 10 dB less Signal to Noise Ratio (SNR) to obtain a Bit Error Rate (BER) similar to its counterpart based on Fourier Transform (FT) in the two-ray channel. It requires about 3 dB less SNR in the channel with frequency offset.

The rest of this paper is organized as follows: In section II, we introduce fractional Fourier transform. Section III introduces FrFT-OFDMA system. MUI analysis is presented in section IV. Section V discusses methods to reduce MUI. Simulation results are presented in section VI. Section VII concludes the paper.

II. FRACTIONAL FOURIER TRANSFORM

Fractional Fourier transform, introduced in [18], is a powerful tool to analyze time varying signals as it preserves signal properties in both time and frequency domain [19]. Let us consider the time-frequency plane where the horizontal and vertical axes represent the time domain and frequency domain, respectively. The traditional Fourier transform can be interpreted as a counterclockwise rotation of coordinates with angle of $\frac{\pi}{2}$ which converts signals from the horizontal to vertical axis. The FrFT, on the other hand, is defined by a rotation of coordinates in time-frequency plane with arbitrary angle α as follows [20].

$$S_{\alpha}(f) = \begin{cases} \int_{-\infty}^{\infty} C_{\alpha} \exp\left\{-\frac{jtf}{\sin\alpha} + \frac{j}{2}(t^2 + f^2)\cot\alpha\right\} s(t) dt & \alpha \neq 2k, k \in \mathbb{Z} \\ s(f) & \alpha = 4k, k \in \mathbb{Z} \\ s(-f) & \alpha = 4k + 2, k \in \mathbb{Z} \end{cases} \quad (1)$$

Where $C_{\alpha} = \frac{\exp\left(-\frac{j}{2}\left(\frac{\pi}{2}\text{sgn}(\sin\alpha) - \alpha\right)\right)}{\sqrt{2\pi|\sin\alpha|}}$, $\alpha = \frac{\pi}{2}a$ and \mathbb{Z} is the set of integer numbers. $s(\cdot)$ and $S_{\alpha}(\cdot)$ are the signal in the time domain and the FrFT domain with parameter α , respectively. As presented in (1), the basis functions of this transform are chirp functions with rate $-\cot\alpha$. The above relationship indicates that FT is a special case of FrFT when $\alpha = \frac{\pi}{2}$. As mentioned earlier, several forms for discrete FrFT have been proposed. The following relationship is a commonly used DFrFT in which the discrete time domain signal and the transformed signal are presented by $s(n)$ and $S_{\alpha}(m)$ respectively [10]

$$S_{\alpha}(m) = \begin{cases} B_{\alpha} \exp\left(\frac{j}{2}\cot(\alpha)m^2\Delta u^2\right) \sum_{n=0}^{M-1} \exp\left(-\frac{j\text{sgn}(\sin\alpha)2\pi nm}{M} + \frac{j}{2}\cot(\alpha)n^2\Delta t^2\right) s(n) & \alpha \neq 2k \\ s(m) & \alpha = 4k \\ s(-m) & \alpha = 4k + 2 \end{cases} \quad (2)$$

where $0 \leq m \leq M - 1$, $k \in \mathbb{Z}$, $B_\alpha = \sqrt{\frac{|\sin\alpha| - j\text{sgn}(\sin\alpha)\cos\alpha}{M}}$, M denotes the length of the DFrFT block, and Δt and Δu are the sampling intervals in time and transform domain, respectively. To have a unitary and invertible transform, Δt and Δu should satisfy the following equation

$$\Delta t \Delta u = \frac{\text{sgn}(\sin\alpha) 2\pi}{M} \sin\alpha \quad (3)$$

Inverse FrFT and inverse discrete FrFT are obtained by substituting $-\alpha$ instead of α in (1) and (2), respectively.

III. SYSTEM MODEL

In this section, we develop the fundamentals of a multi-user transmission scheme based on FrFT. Since the block diagram of this system is analogous to that of OFDMA where FT is replaced by FrFT, we call it FrFT-OFDMA. In what follows, we assume that the FrFT-OFDMA is used for downlink transmission. We also present the original data block length, the total number of sub-carriers and the number of users by N , M and K , respectively.

Let \mathbf{d}_k denote the original data block of the k -th user. Exploiting the mapping matrix \mathbf{T}_k , N out of total M sub-carriers are assigned to the k -th user. The mapping strategies are commonly categorized to localized and interleaved mapping. In the former, adjacent sub-carriers are allocated to one user while in the latter, to obtain frequency diversity, the sub-carriers assigned to one user are distanced with K sub-carriers. In localized mapping, the element of the m -th row and n -th column of \mathbf{T}_k is expressed as follows

$$T_k(m, n) = \delta(m - n - (k - 1)N) \quad (4)$$

where $0 \leq n \leq N - 1$, $0 \leq m \leq M - 1$ and $\delta(\cdot)$ is the Delta Dirac function. The components of the mapping matrix in interleaved scheme is

$$T_k(m, n) = \delta(m - nK - k) \quad (5)$$

Assuming the downlink transmission, the signal of all users are passed through an IFrFT block and added together to generate the output signal as follows

$$\mathbf{s} = \sum_k \mathbf{F}_{M, -\alpha_k} \mathbf{T}_k \mathbf{d}_k \quad (6)$$

where $\mathbf{F}_{M, -\alpha_k}$ presents the M point IFrFT matrix with angle α_k . As will be explained later, this matrix

is not necessarily similar for all users as α_k differs for distinct users. To obtain a better insight into the

FrFT-OFDMA system, let us compare the sub-carriers of this system with those of OFDMA. For a system with block duration of T , the distance between adjacent sub-carriers is $\frac{1}{T}$. Keeping this in mind,

the frequency of the k -th user's sub-carriers in localized OFDMA is from $\frac{(k-1)N}{T}$ to $\frac{kN-1}{T}$. The corresponding values for FrFT-OFDMA is from $\frac{(k-1)N\sin\alpha_k}{T}$ to $\frac{(kN-1)\sin\alpha_k}{T}$ in the fractional Fourier domain with angle α_k while their frequencies vary with time by rate $-\cot\alpha_k$.

The received signal at the u -th Mobile Station (MS) can be written as

$$\mathbf{r}_u = \mathbf{H}_u \sum_k \mathbf{F}_{M,-\alpha_k} \mathbf{T}_k \mathbf{d}_k + \mathbf{n} \quad (7)$$

where \mathbf{H}_u denotes the channel matrix between the Base Station (BS) and the u -th MS and \mathbf{n} shows the additive white Gaussian noise vector. At the receiver, using an M -point FrFT with angle α_u the signal is converted into transform domain to be equalized. After equalization in the transform domain, the signal can be expressed as

$$\mathbf{Z}_u = \mathbf{E}_u \mathbf{F}_{M,\alpha_u} \mathbf{H}_u \sum_k \mathbf{F}_{M,-\alpha_k} \mathbf{T}_k \mathbf{d}_k + \mathbf{E}_u \mathbf{F}_{M,\alpha_u} \mathbf{n} \quad (8)$$

where the diagonal matrix, \mathbf{E}_u , denotes the single tap equalizer of the u -th user whose elements are derived as [21]

$$\mathbf{E}_u(k, k) = \frac{\tilde{h}_u^*(k, k)}{\sum_{i=0}^{N-1} \tilde{h}_u(k, i) \tilde{h}_u^*(k, i) + E\{\eta_{\alpha_u}(k) \eta_{\alpha_u}^*(k)\}} \quad (9)$$

where $\eta_{\alpha_u} = \mathbf{F}_{M,\alpha_u} \mathbf{n}$, $\tilde{h}_u(m, n)$ presents the element of the m -th row and n -th column of the matrix $\tilde{\mathbf{H}}_u = \mathbf{F}_{M,\alpha_u} \mathbf{H}_u \mathbf{F}_{M,-\alpha_u}$ and $E\{\cdot\}$ denotes the expectation. Finally, after demapping the signal can be written as

$$\begin{aligned} \tilde{\mathbf{d}}_u &= \mathbf{T}_u^T \mathbf{E}_u \mathbf{F}_{M,\alpha_u} \mathbf{H}_u \sum_k \mathbf{F}_{M,-\alpha_k} \mathbf{T}_k \mathbf{d}_k + \mathbf{T}_u^T \mathbf{E}_u \mathbf{F}_{M,\alpha_u} \mathbf{n} \\ &= \underbrace{\mathbf{D} \mathbf{d}_u}_{\text{Desired signal}} + \underbrace{(\mathbf{T}_u^T \mathbf{E}_u \mathbf{F}_{M,\alpha_u} \mathbf{H}_u \mathbf{F}_{M,-\alpha_k} \mathbf{T}_k - \mathbf{D}) \mathbf{d}_u}_{\text{ICI}} + \underbrace{\sum_{k \neq u} \mathbf{T}_u^T \mathbf{E}_u \mathbf{F}_{M,\alpha_u} \mathbf{H}_u \mathbf{F}_{M,-\alpha_k} \mathbf{T}_k \mathbf{d}_k}_{\text{MUI}} \end{aligned} \quad (10)$$

where $\mathbf{D} = \text{diag}(\mathbf{T}_u^T \mathbf{E}_u \mathbf{F}_{M,\alpha_u} \mathbf{H}_u \mathbf{F}_{M,-\alpha_k} \mathbf{T}_k)$ and $(\cdot)^T$ presents transpose of a matrix. According to

(10), the demapped signal consists of the desired signal, ICI and MUI.

IV. MULTI-USER INTERFERENCE ANALYSIS

As stated earlier, the optimum angle of transform for each user depends on its channel impulse response. Thus, in a wireless scenario, different users do not have similar angles of transform necessarily. This can pose critical challenges as the orthogonality of sub-carriers can be violated. In this section, we investigate the multi-user interference generated by the difference of α_k 's. To pinpoint only the effect of non-equal angles of transform on MUI and make sure the received signal is not

affected by any other type of interference, in this section we analyze the system in Additive White Gaussian Noise (AWGN) channel.

The FrFT-OFDMA can be considered as a GMC system whose sub-carriers are chirp functions with rate $-cot\alpha$. Such a chirp function with original frequency of $\frac{k}{T}$ can be written as

$$f_{\alpha,k}(t) = \sqrt{\frac{\sin\alpha + j\cos\alpha}{T}} \exp\left(-\frac{j}{2}\left(t^2 + \left(\frac{2\pi k \sin\alpha}{T}\right)^2\right)cot\alpha + \frac{jk2\pi}{T}t\right) \quad (11)$$

The inner product of two chirp functions with rates $-cot\alpha_1$ and $-cot\alpha_2$ and original frequencies of $\frac{m}{T}$ and $\frac{n}{T}$ is

$$\int_{-\infty}^{\infty} f_{\alpha_1,m}(t)f_{\alpha_2,n}^*(t)dt = e^{\frac{j(\alpha_2-\alpha_1)}{2}} \exp\left(\frac{j}{4}\left(\frac{2\pi}{T}\right)^2(n^2 \sin(2\alpha_2) - m^2 \sin(2\alpha_1))\right) \\ \times \int_0^T \exp\left(\frac{j}{2}t^2(cot\alpha_2 - cot\alpha_1)\right) \exp\left(-\frac{j2\pi}{T}(n-m)t\right) dt \quad (12)$$

If the chirp functions have identical rates and their original frequencies are distanced by a multiple of $\frac{1}{T}$, the chirp functions are orthogonal. From another perspective, a chirp signal with rate $-cot\alpha$ is a Dirac delta function in the fractional Fourier domain with angle α . However, if the angle of transform differs from α , the transformed signal is not a delta function. To calculate the MUI in an FrFT-OFDMA system with multiple angles of transform in an AWGN channel, let us consider the output data of the u -th receiver which can be expressed as

$$\tilde{\mathbf{d}}_u = \mathbf{d}_u + \underbrace{\sum_{k \neq u} \mathbf{T}_u^T \mathbf{F}_{M,\alpha_u} \mathbf{F}_{M,-\alpha_k} \mathbf{T}_k \mathbf{d}_k}_{MUI} + \mathbf{n} \quad (13)$$

Regarding (2) and (3), we have

$$[\mathbf{F}_{M,\alpha_u} \mathbf{F}_{M,-\alpha_k}](m,n) = \frac{\exp\left(\frac{j(\alpha_u - \alpha_k)}{2}\right)}{M} \exp\left(\frac{j}{4}\left(\frac{2\pi}{M\Delta T}\right)^2(m^2 \sin(2\alpha_u) - n^2 \sin(2\alpha_k))\right) \\ \times \sum_{l=0}^{M-1} \exp\left(\frac{j}{2}\Delta T^2 l^2(cot\alpha_u - cot\alpha_k)\right) \exp\left(-\frac{j2\pi l}{M}(m-n)\right) \quad (14)$$

Therefore, the data of the l -th sub-carrier of the u -th user is detected as

$$\tilde{\mathbf{d}}_u(i) = \mathbf{d}_u(i) + \sum_{k \neq u} \sum_{n=0}^{N-1} \mathbf{d}_k(n) \frac{\exp\left(\frac{j(\alpha_u - \alpha_k)}{2}\right)}{M} \\ \times \exp\left(\frac{j}{4}\left(\frac{2\pi}{M\Delta T}\right)^2((i + (u-1)N)^2 \sin(2\alpha_u) - (n + (k-1)N)^2 \sin(2\alpha_k))\right) \\ \times \sum_{l=0}^{M-1} \exp\left(\frac{j}{2}\Delta T^2 l^2(cot\alpha_u - cot\alpha_k)\right) \exp\left(-\frac{j2\pi l}{M}(i - n + (u-k)N)\right) \quad (15)$$

which consists of the desired part $\mathbf{d}_u(i)$ and an interference generated by other users. The MUI power is derived as

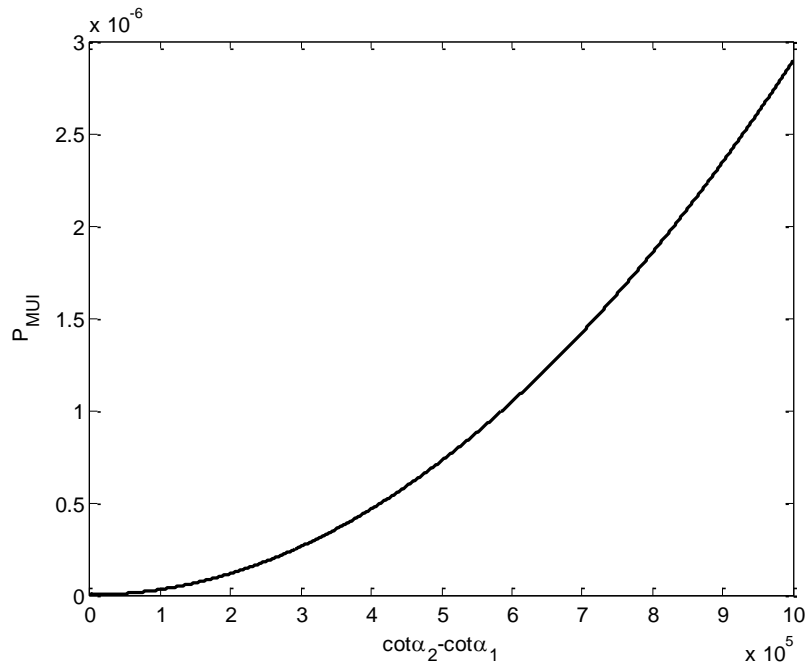


Fig. 1. P_{MUI} versus $cota_2 - cota_1$ for an FrFT-OFDMA system with two users where $M = 128$ and $\Delta T = 10^{-6}$.

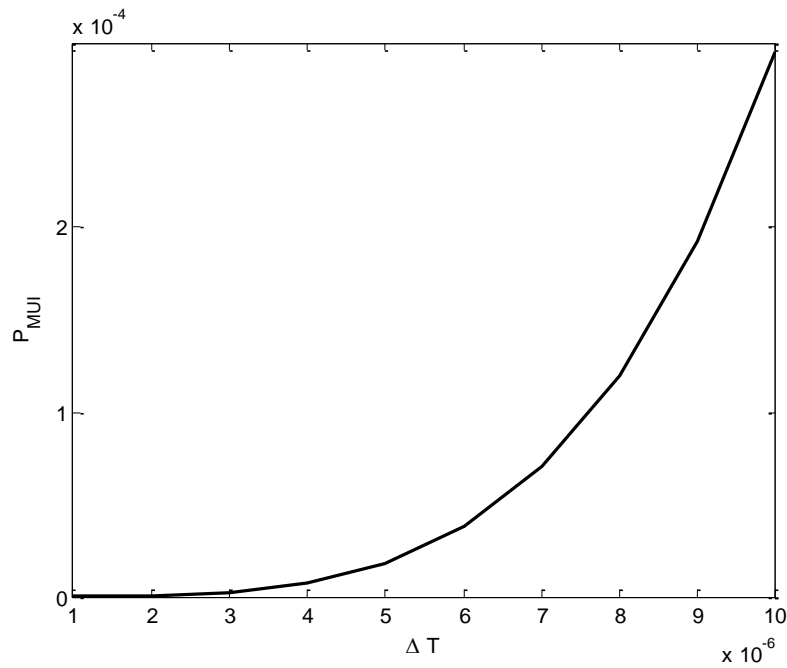


Fig. 2. P_{MUI} versus ΔT for an FrFT-OFDMA system with two users where $M = 128$ and $cota_2 - cota_1 = 10^5$.

$$P_{MUI} = \sum_{k \neq u} \sum_{n=0}^{N-1} \frac{1}{M^2} \left| \sum_{l=0}^{M-1} \exp \left(-\frac{j2\pi l}{M} \left(i - n + (u - k)N - \frac{M}{4\pi} \Delta T^2 l (cota_u - cota_k) \right) \right) \right|^2 \quad (16)$$

Based on (16), if $cota_u = cota_k$ the MUI power is zero. Otherwise, the MUI power increases by

$\cot\alpha_u - \cot\alpha_k$, M or ΔT as presented in Fig. 1 and Fig. 2. Both figures show the MUI power for an FrFT-OFDMA system with two users and $M = 128$. In Fig. 1 where the effect of the difference of cotangent of angles of transform on MUI power is presented, ΔT is assumed 10^{-6} . Fig. 2 depicts P_{MUI} versus ΔT for a system with $\cot\alpha_2 - \cot\alpha_1 = 10^5$.

V. METHODS TO REDUCE MUI

In this section, we investigate the conditions under which multi-user interference is tolerable and propose solutions whereby reduce MUI.

A. Quasi-Orthogonally Conditions

Regarding (12), two sub-carriers are orthogonal if their instantaneous frequencies are distanced by multiple of $\frac{1}{T}$. The instantaneous frequency of a chirp function with rate $-\cot\alpha$ and original frequency of $\frac{k}{T}$ is $f(t) = \frac{k}{T} - \frac{t\cot\alpha}{2\pi}$ [8]. Let us define $\Delta f_{i,j}$ as the difference between the instantaneous frequencies of two chirp functions with rates $-\cot\alpha_i$ and $-\cot\alpha_j$ and original frequencies of $\frac{k_i}{T}$ and $\frac{k_j}{T}$, respectively. Then,

$$\Delta f_{i,j}(t) = \frac{k_i - k_j}{T} - \frac{t(\cot\alpha_i - \cot\alpha_j)}{2\pi} \quad (17)$$

If $\cot\alpha_i - \cot\alpha_j$ is small enough, the power of multi-user interference between these users is negligible. Thus, the performance of FrFT-OFDMA is analogous to that of OFDMA in AWGN. We define this condition as quasi-orthogonality.

Fig. 3 presents the BER of OFDMA and quasi-orthogonal FrFT-OFDMA systems in an AWGN channel. The curves are generated for the first user of a 2-user system with $M = 128$, $\Delta T = 10^{-4}$, $\cot\alpha_1 = 0$ and $\cot\alpha_2 = 636.62$. Since the users are quasi-orthogonal the BER performance of the FrFT-OFDMA system is almost identical to that of OFDMA.

B. Spectral Guard Band

Non-orthogonality of sub-carriers resides in the fact that the fractional Fourier transform of a chirp function is not a delta function if the chirp rate and angle of transform do not match. If the mismatch between these parameters is significant enough to violate the quasi-orthogonally condition, the sub-carriers allocated to one user drift and contaminate the bordering sub-carriers of adjacent users. Thus, the performance of the system significantly decreases due to multi-user interference. It is noteworthy that the impact of this phenomenon on interleaved mapping is more significant than on localized mapping since in the former the neighboring sub-carriers belong to different users and consequently

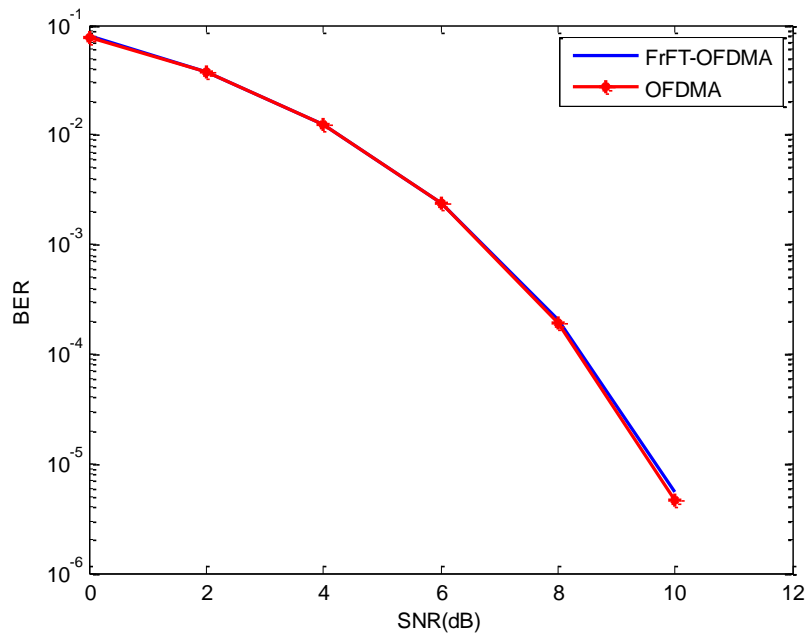


Fig. 3. Performance comparison of a 2-user OFDMA and quasi-orthogonal FrFT-OFDMA system in AWGN channel with $M = 128$, $\Delta T = 10^{-4}$, $cot\alpha_1 = 0$ and $cot\alpha_2 = 636.62$.

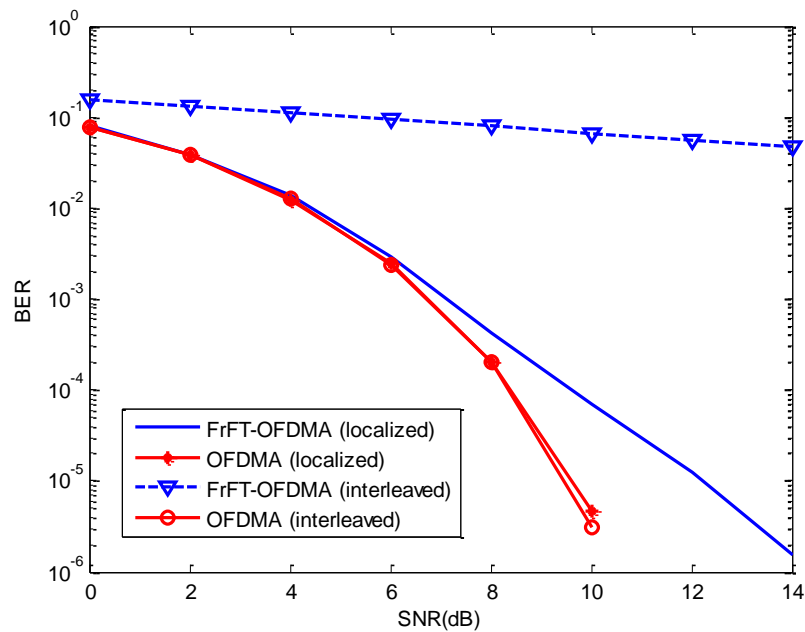


Fig. 4. Performance comparison of a 2-user OFDMA and non-orthogonal FrFT-OFDMA system in AWGN channel with different mapping schemes and $M = 128$, $\Delta T = 10^{-4}$, $cot\alpha_1 = 636.62$ and $cot\alpha_2 = 6.366 \times 10^4$.

all sub-carriers are affected by the interference. The performance loss due to MUI is presented in Fig. 4 for the first user of a 2-user FrFT-OFDMA and OFDMA system where $M = 128$, $\Delta T = 10^{-4}$, $cot\alpha_1 = 636.62$ and $cot\alpha_2 = 6.366 \times 10^4$. As discussed earlier and presented in this figure, interleaved mapping is not appropriate for FrFT-OFDMA systems. Hence, we will concentrate on

localized mapping hereafter. For the system under discussion in Fig. 4, the sub-carriers 0 to 63 and 64 to 127 are assigned to the first user and second user in their corresponding FrFT-domain. The MUI of each user contaminates two out-of-band sub-carriers. We can overcome this issue by embedding two null sub-carriers as spectral guard interval between the sub-carriers allocated to different users. To efficiently reduce MUI, the length of the guard band has to be selected such that $\Delta f_{i,j}$, defined in (17), turns to a multiple of $\frac{1}{T}$. Since $\Delta f_{i,j}$ varies with time, we consider the maximum guard band derived for $t = T$. Let us define

$$L_{i,j} = \frac{T^2(\cot\alpha_i - \cot\alpha_j)}{2\pi} \quad (18)$$

Based on the above equation, we experimentally define a threshold of $\frac{1}{10}$ to quantify the quasi-orthogonally condition. In other words, if $L_{i,j} < \frac{1}{10}$, we consider the users quasi-orthogonal and do not embed any guard interval. Otherwise, we use a guard interval whose length is

$$G_{i,j} = [L_{i,j}]u(L_{i,j} - 0.1) \quad (19)$$

where $[x]$ is the smallest integer number greater than x and $u(\cdot)$ is the unit step function. Therefore,

the first term in $\Delta f_{i,j} = \frac{k_i - k_j + G_{i,j}}{T} - \frac{t(\cot\alpha_i - \cot\alpha_j)}{2\pi}$ is dominant and we can approximate

$\Delta f_{i,j} \cong \frac{k_i - k_j + G_{i,j}}{T}$. In this system the total number of sub-carriers is $M = KN + \sum G_{i,j}$.

The performance of the proposed scheme is presented in Fig. 5. The parameters, except for the spectral guard band, are similar to those of Fig. 4. As can be seen, although the users are not orthogonal or quasi-orthogonal, using a guard band we are able to improve the BER of FrFT-OFDMA such that it gets sufficiently close to that of OFDMA.

For an OFDMA symbol with duration T and KN subcarriers (K users and N subcarriers for each user) the bandwidth required is $\frac{KN}{T}$. Since the instantaneous frequency of a chirp function with rate $-\cot\alpha$ and original frequency of $\frac{k}{T}$ is $f(t) = \frac{k}{T} - \frac{t\cot\alpha}{2\pi}$, the bandwidth utilized by a chirp signal with duration T and chirp rate of $\cot\alpha$ is $\frac{T\cot\alpha}{2\pi}$. So the bandwidth required in our proposed FrFT-OFDMA scheme is $\frac{KN+G}{T} + \frac{T\cot\alpha_1}{2\pi}u(\cot\alpha_1) + \frac{T\cot\alpha_K}{2\pi}u(-\cot\alpha_K)$, where G is the number of subcarriers inserted as guard band, α_1 and α_K are the FrFT order of the first and last user, respectively and $u(\cdot)$ is the unit step function. If the users are orthogonal or quasi-orthogonal G is zero. So our proposed scheme requires $\frac{G}{T} + \frac{T\cot\alpha_1}{2\pi}u(\cot\alpha_1) + \frac{T\cot\alpha_K}{2\pi}u(-\cot\alpha_K)$ more bandwidth than OFDMA for a fixed bit rate. Some numerical examples of increase in the bandwidth are given in section VI.

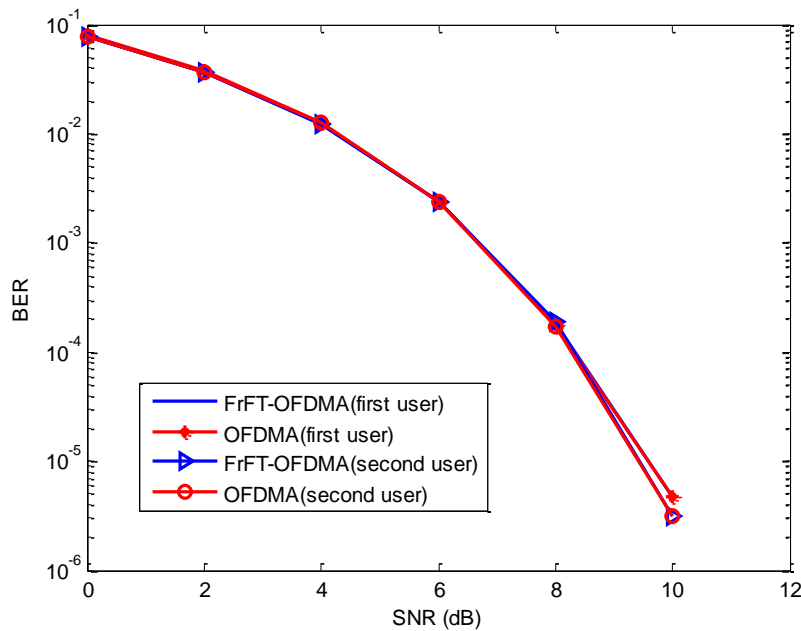


Fig. 5. Performance comparison of a 2-user OFDMA and non-orthogonal FrFT-OFDMA system in AWGN channel with spectral guard band and $M = 128$, $\Delta T = 10^{-4}$, $\cot\alpha_1 = 636.62$ and $\cot\alpha_2 = 6.366 \times 10^4$.

VI. SIMULATION RESULTS

In this section, we present the performance of our proposed multi-user scheme based on fractional Fourier transform in two types of channels, a time invariant channel with frequency offset, and a two-ray doubly dispersive channel.

A. Time Invariant Channel with Frequency Offset

Frequency offset, which occurs due to oscillator's frequency instabilities, can significantly degrade the performance of OFDMA system [22]. In the presence of frequency offset, the channel matrix will no longer be diagonal which generates ICI and MUI. In what follows, we show how selection of an appropriate angle of transform in an FrFT-OFDMA system, reduces this effect.

We consider a system with four users, $M = 128$ and $\Delta T = 10^{-6}$. The data blocks are transmitted over a channel whose power-delay profile is presented in Table I. The frequency offset of each user is determined in Table II. Knowing the channel of each user, we can determine the user's optimum angle of transform. The optimum angle of transform is determined by the angle which minimizes the Mean Square Error (MSE) between the transmitted and received data symbols. This MSE is presented versus $\cot\alpha$ for the third user of the considered FrFT-OFDMA system in Fig. 6. To show the effect of angle of transform, we present the BER performance of FrFT-OFDMA (with optimum angle of transform) and OFDMA systems in Fig. 7. Although only the BER of the second and third users are

TABLE I. CHANNEL POWER DELAY PROFILE

Path gains (dB)	0	-2.171	-4.342	-6.514	-8.685	-10.85	-13.02	-15.20	-17.37	-19.54
Path delays (μsec)	0	4	8	12	16	20	24	28	32	36

TABLE II. FREQUENCY OFFSET OF THE USERS

User number	Normalized frequency offset
1	0.08
2	0.1
3	0.12
4	0.15

depicted, the performance of other users has a similar manner. For a fixed bit rate, the FrFT-OFDMA system in this example requires only **0.22%** more bandwidth than the OFDMA system but as can be seen, it outperforms OFDMA, significantly.

B. Two-Ray Doubly Dispersive Channel

As stated earlier, OFDMA loses performance in doubly dispersive channels since the exponential sub-carriers no longer match the channels singular functions. Although in general it is difficult to determine the singular functions of a time varying channel, for specific cases they can be determined theoretically. For instance, the singular functions of a channel with a linear spread function in the delay-Doppler plane are chirp functions [23], [24]. In other words, a channel with a spread function of

$$S(\nu, \tau) = g(\tau)\delta(\nu - \mu\tau) \quad (20)$$

has singular functions as

$$v_i(t) = u_i(t) = \exp(j\pi\mu t^2) \exp(j2\pi f_i t) \quad (21)$$

where ν , τ and μ depict the Doppler frequency, path delay and the slope of the spread function in the delay-Doppler plane, respectively. Hence, the FrFT with an appropriate angle of transform is optimum for transmission over channels whose Doppler frequency and delay of each path are linearly related as

$$\mu = \frac{f_0}{\tau_0} = \frac{f_1}{\tau_1} = \dots = \frac{f_L}{\tau_L} \quad (22)$$

where f_i , τ_i and L are the i -th path Doppler frequency, i -th path delay and the number of paths, respectively.

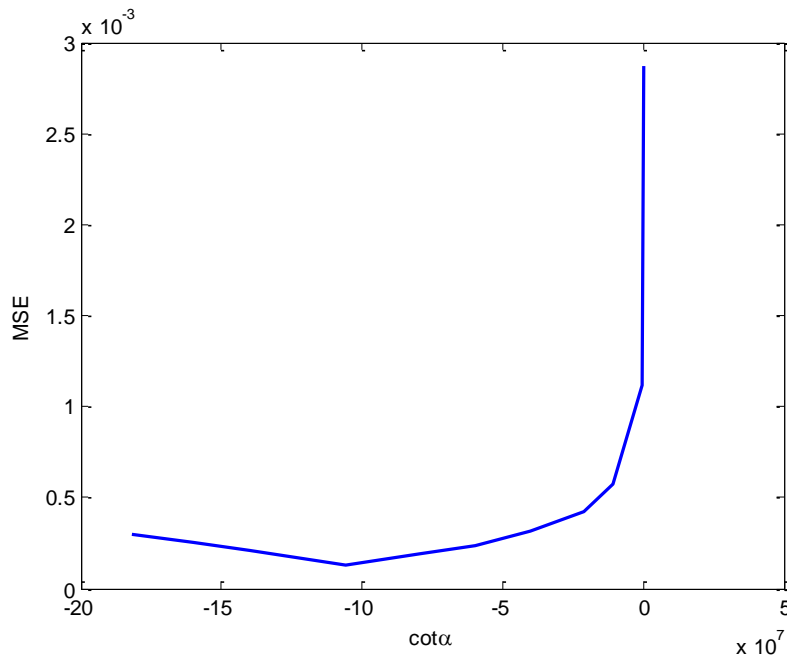


Fig. 6. MSE versus cotangent of angle of transform for third user of a 4-user FrFT-OFDMA system with $M = 128$ and $\Delta T = 10^{-6}$ in a channel with parameters of table I and II.

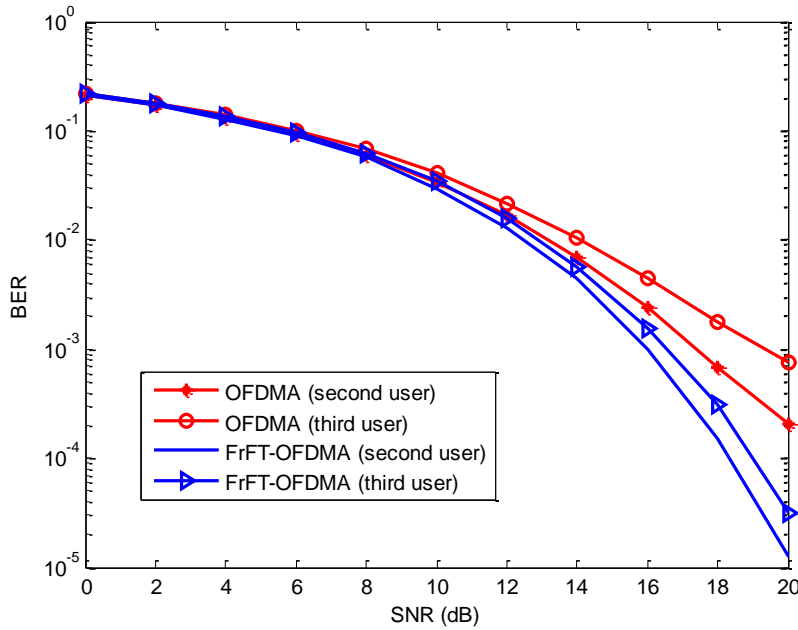


Fig. 7. Performance of 4-user FrFT-OFDMA and OFDMA systems in the presence of frequency offset with $M = 128$ and $\Delta T = 10^{-6}$ in a channel with parameters of table I and II.

This is presented as the line $v = \mu\tau$ in the delay-Doppler plane. In practice, a channel with a linear spread function in this plane is a two-ray channel whose spread function is expressed as

$$S(v, \tau) = h_0\delta(\tau - \tau_0)\delta(v - f_0) + h_1\delta(\tau - \tau_1)\delta(v - f_1) \quad (23)$$

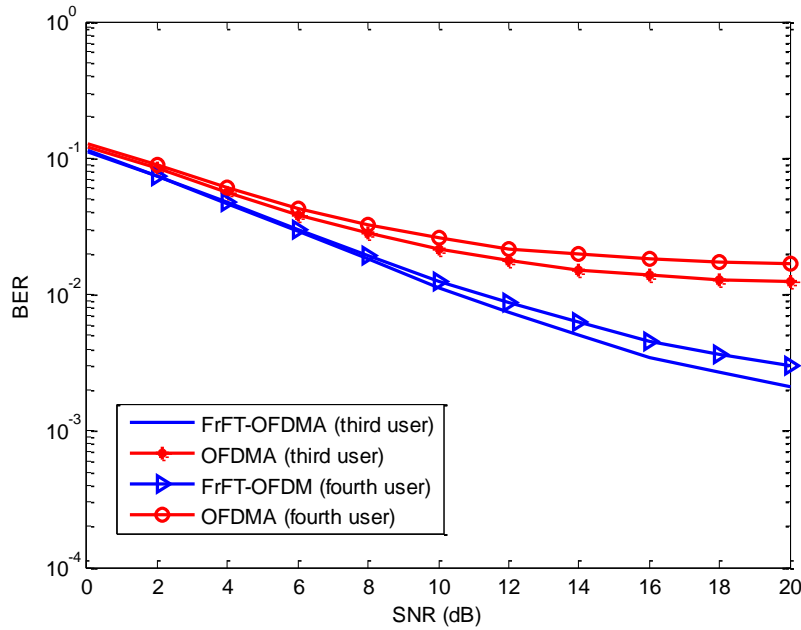


Fig. 8. Performance of 4-user FrFT-OFDMA and OFDMA systems in a two-ray channel with power delay profile of table III and $M = 128$ and $\Delta T = 10^{-6}$.

TABLE III. POWER-DELAY PROFILE OF THE TWO-RAY CHANNEL

Path gain (dB)	0	-4.3429
Path delay (μsec)	0	9

where τ_0 and τ_1 denote the delays and f_0 and f_1 are the Doppler frequencies of the first and second path, respectively. To match the FrFT with the channel singular functions, the chirp rate has to be the same as the time variation rate of the singular functions. In other words, $\mu = \frac{\cot\alpha}{2\pi}$. Equivalently, for a channel with a linear spread function in delay-Doppler plane, the optimum angle of transform is $\alpha = \cot^{-1}\left(\frac{2\pi f_1}{\tau_1}\right)$. It is noteworthy that the amplitude of each channel path might be constant or time varying. For both cases, the derived α minimizes the MSE. However, in the former case, as opposed to the latter, it also diagonalizes the channel matrix and cancels the ICI completely.

For a general multi-path channel, if the spread function is concentrated around a line in the delay-Doppler plane, we can approximate it by a line and derive the angle of transform accordingly. For channels where we have

$$v = \mu\tau + f_{\text{intercept}} \quad (24)$$

the optimum angle of transform is determined using the MMSE criterion between the transmitted and received data.

Fig. 8 depicts the BER of the proposed FrFT-OFDMA system in a two-ray channel with linear relation of $\nu = \mu\tau$ between the delays and Doppler frequencies of the paths. The figure presents the BER of the third and fourth users in the system where total number of users is four. The power-delay profile of this channel is represented in Table III.

We have also assumed that the amplitude of the channel paths vary with time based on the Jakes model [25]. In this system, according to (18), we have $L_{1,2} = 0.1717$, $L_{2,3} = 0.2579$ and $L_{3,4} = 0.2577$. Therefore, we embed one null sub-carrier between the sub-carriers of neighboring users. The FrFT-OFDMA system utilizes 2.54% more bandwidth than the OFDMA system. But as can be seen, the system based on FrFT outperform that based on the traditional Fourier transform with at least 10 dB improvement at BER of 10^{-2} .

VII. CONCLUSION

In this paper, to improve the performance of OFDMA system in doubly dispersive channels, a novel multiple access scheme, named FrFT-OFDMA, was proposed. This system utilizes FrFT instead of FFT while the angle of transform is selected to minimize the ICI. Thus, the optimum angle of transform, which depends on the user's channel parameters, is different for distinct users. This violates the orthogonality of users and generate MUI. By MUI analysis, we showed that MUI power increases as the difference between the cotangent of angles of transform increases. We defined quasi-orthogonality as a condition where this difference and consequently the MUI power is negligible. For non-orthogonal and non-quasi-orthogonal users, we suggested to use a spectral guard interval between adjacent users to cancel MUI. We examined the performance of the proposed scheme over a time invariant channel with frequency offset and a two-ray double dispersive channel. We showed that FrFT-OFDMA significantly outperform its counterpart based on Fourier transform such that we obtain about 3 dB gain for the first channel and about 10 dB gain for the second channel.

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