Steady-State Performance Analysis of Incremental Variable Tap-Length Algorithm in WSNs Under Noisy Links condition

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***Abstract*- Recently proposed distributed incremental fractional tap-length (FT) variable-length least mean square (LMS) technique do not consider noisy links errors, which occur during the communication of local estimations between nodes. In this paper, we study the noisy links effect on the performance of this algorithm. We derive a mathematical formulation for the steady-state length at each node. Our derived relationship shows how steady-state tap-length is affected by noisy links. Simulations confirm that there is a good match between the theory and simulated tap-length. Furthermore, the critical result is that, as the noise level increases, the steady-state tap-length decreases compared to the ideal link version. However, in low noise conditions, this length is still larger than the optimal filter length.**

Index Terms- Adaptive networks, distributed estimation, fractional tap-length, noisy links, steady-state tap-length.

I. INTRODUCTION

A wireless sensor network (WSN) consists of distributed sensors that cooperate to estimate and track the desired parameter, such as target position, average temperature, etc. In general, such a task in WSNs can be performed by either a centralized strategy or a decentralized one [1]. A distributed approach is scalable concerning both communication resources and computational power. In this strategy, information distributes through the network, and each sensor participates in the estimating task. Networks that perform the processing task with distributed adaptive estimation algorithms [2]-[7] are called adaptive networks. Employing cooperative adaptive processing facilitates the tracking of both environmental and network topology variations.

Various distributed adaptive techniques have been reported in the literature. These methods classify based on the cooperation strategy between sensors (diffusion and incremental) and the adaptive algorithm they use. The incremental cooperation requires a cyclic path through the network, and all sensors communicate with their neighbors within this path. This cooperation strategy reduces the

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required power and communication resources. While, in diffusion cooperation, each sensor communicates with all of its neighboring nodes as dictated by the network topology. This strategy is proper when more communication and power resources are available.

In the reported works [2]-[7], the length of the adaptive filter at each sensor is considered to be fixed, which is not proper for some conditions. The number of tap coefficients is a critical parameter that significantly affects the performance of an adaptive algorithm. In the stand-alone filter domain, several variable tap-length algorithms are available that attempt to accelerate the convergence of the least mean square (LMS) technique [8]-[10]. This is not the only motivation for the variable tap-length adaptive approaches. They are also considered in conditions where tap-length is unknown or even variable. So, structure adaption algorithms have been proposed. Many algorithms have been introduced in the stand-alone filter domain for this purpose [11]-[22]. The most practical technique among them is the fractional tap-length (FT) algorithm [22]. This algorithm, is simple and yet has shown a good performance. Therefore, it is suitable to be considered as a popular algorithm. So, it is no surprise that this algorithm has only been considered for structure adaption in the adaptive networks domain among all variable tap-length techniques. Reference [23] has been developed the FT algorithm with the incremental strategy for distributed networks. In [23], the data have been assumed to be communicated between sensors without any distortion. This assumption, however, may not be logical in practical situations due to the link noises. Here, we analyze the impact of noisy links on the performance of the distributed incremental FT variable tap-length LMS algorithm. We present an analysis for this algorithm in the incremental distributed networks with noisy links. According to the performed analysis, an expression for the steady-state tap-length at each node is derived. Our study shows how the steady-state tap length is affected by noisy links. Nevertheless, the critical result is that, as the noise level increases, the steady-state tap-length decreases compared to the ideal link version. However, in low noise conditions, this length is still larger than the optimal filter length. Computer simulations support the theoretical analysis and discussions.

The roadmap of this paper is as follows. In section II, we review the distributed incremental FT variable tap-length LMS algorithm. In section III, we reformulate this algorithm when links between sensors are noisy. In section IV, we analyze the performance of incremental FT variable tap-length LMS algorithm with noisy links and provide a mathematical expression for the steady-state tap-length at each node. Then, we discuss the derived theoretical results. Theoretical results are compared with the simulations in section V, and section VI provides concluding remarks.

II. INCREMENTAL FT-LMS ALGORITHM

We consider a collection of N sensors distributed over some areas. The primary purpose is to estimate both the length L_{opt} and coefficients of the desired vector $w^{o}_{L_{opt}}$ from collected measurements at sensors, i.e., from time realizations $\{d_k(i), \boldsymbol{u}_{k,i}\}$ of zero-mean spatial data $\{\mathrm{d}_k, \mathbf{u}_k\}$, where each $\mathbf{u}_k$ is a $1\times L$ row regression vector, and each d_k is a scalar measurement.

Let $\boldsymbol{\psi}_k^{(i)}$ be a local estimate for the unknown vector at sensor *k* and time *i*. In [2], a distributed incremental LMS (DILMS) strategy for estimation of the desired vector coefficients has been presented as follows:

$$\boldsymbol{\psi}_k^{(i)} = \boldsymbol{\psi}_{k-1}^{(i)} + \mu_k \boldsymbol{u}_{k,i}^{*}\left(d_k(i) - \boldsymbol{u}_{k,i}\boldsymbol{\psi}_{k-1}^{(i)}\right), \qquad k \in \mathrm{N} \tag{1}$$

where μ_k is the local step-size parameter.

In [23], a solution has been proposed to estimate the unknown vector length by extending the FT algorithm within the incremental strategy. With the assumption that $\ell_{k,f}(i)$ indicates the local estimation of the fractional tap-length at sensor *k* at time *i*, in this algorithm, the sensor *k* updates its local estimate as [23]:

$$\ell_{k,f}(i) = \left(\ell_{k-1,f}(i) - a_k\right) - \gamma_k\left[\left(e_{k,L_k(i)}^{(L_k(i))}(\boldsymbol{\psi}_{k-1}^{(i)})\right)^2 - \left(e_{k,L_k(i)-\Delta}^{(L_k(i))}(\boldsymbol{\psi}_{k-1}^{(i)})\right)^2\right] \tag{2}$$

where

$$e_{k,L_k(i)}^{(L_k(i))}\left(\boldsymbol{\psi}_{k-1}^{(i)}\right) = d_k(i) - \boldsymbol{u}_{k,i}\boldsymbol{\psi}_{k-1}^{(i)} \tag{3}$$

and

$$e_{k,L_k(i)-\Delta}^{(L_k(i))}\left(\boldsymbol{\psi}_{k-1}^{(i)}\right) = d_k(i) - \boldsymbol{u}_{k,i}(1:L_k(i)-\Delta)\boldsymbol{\psi}_{k-1}^{(i)}(1:L_k(i)-\Delta) \tag{4}$$

in which $\boldsymbol{u}_{k,i}(1:L_k(i)-\Delta)$ and $\boldsymbol{\psi}_{k-1}^{(i)}(1:L_k(i)-\Delta)$ consist of the initial $L_k(i)-\Delta$ entries of $\boldsymbol{u}_{k,i}$ and $\boldsymbol{\psi}_{k-1}^{(i)}$ respectively, where Δ is an integer which prevents the length to be suboptimal. In (2), a_k denotes the local leakage factor and γ_k is the local step-size for $\ell_{k,f}(i)$ adaption at node *k*. The length $\ell_{k,f}(i)$ is no longer considered to be an integer. The local integer length, $L_k(i)$, is computed as [23]:

$$L_{k+1}(i) = \begin{cases} \lfloor \ell_{k,f}(i) \rfloor & \text{if } \left|L_k(i) - \ell_{k,f}(i)\right| \ge \delta_k \\ L_k(i) & \text{otherwise} \end{cases} \tag{5}$$

where δ_k is a small integer, and $\lfloor . \rfloor$ is the floor operator. Also, during the tap-length evolution, the tap-length is forced to be not less than a lower bound $L_{\min}$, where $L_{\min} > \Delta$. The requirement of this operation is due to utilizing $L_k(i)-\Delta$ as a tap-length in the distributed estimation process. With the current tap-length $L_k(i)$, the tap-weights are then updated by (1), where the length of $\boldsymbol{u}_{k,i}$ and $\boldsymbol{\psi}_{k-1}^{(i)}$ are adjusted by $L_k(i)$.

III. EFFECTS OF NOISY LINKS

Under noisy links condition, local estimates of coefficients and filter length, exchanged between neighboring nodes, are affected by link noise. In other words, in the presence of noisy links, instead of receiving $\boldsymbol{\psi}_{k-1}^{(i)}$ and $\ell_{k-1,f}(i)$ in node *k*, the following values are received:

$$\boldsymbol{\psi}_{k-1}^{(i)} + \boldsymbol{q}_{k,i} \tag{6}$$

$$\ell_{k-1,f}(i) + \ell_{q,k,i} \tag{7}$$

The vector $\boldsymbol{q}_{k,i}$ and the scalar $\ell_{q,k,i}$ are the channel noise between sensors $k-1$ and *k* added to the local estimations of unknown parameter and fractional tap-length, respectively. We assume that $\boldsymbol{q}_{k,i}$ is a time realization of a zero-mean random process $\mathbf{q}_k$ with covariance matrix $\mathbf{Q}_k$, and $\ell_{q,k,i}$ is a time realization of a zero-mean random process $\ell_{q,k}$ with variance $\sigma_{c,k}^2$. Considering (6) and (7), the DILMS and the incremental FT update equations can be written as:

$$\boldsymbol{\psi}_k^{(i)} = \boldsymbol{\psi}_{k-1}^{(i)} + \boldsymbol{q}_{k,i} + \mu_k \boldsymbol{u}_{k,i}^* \left[d_k(i) - \boldsymbol{u}_{k,i}(\boldsymbol{\psi}_{k-1}^{(i)} + \boldsymbol{q}_{k,i}) \right] \tag{8}$$

and

$$\ell_{k,f}(i) = \left(\ell_{k-1,f}(i) + \ell_{q,k,i} \right) - a_k - \gamma_k \left[\left(e_{k,L_k(i)}^{(L_k(i))} \right)^2 - \left(e_{k,L_k(i)-\Delta}^{(L_k(i))} \right)^2 \right] \tag{9}$$

where

$$e_{k,L_k(i)}^{(L_k(i))} = d_k(i) - \boldsymbol{u}_{k,i}(\boldsymbol{\psi}_{k-1}^{(i)} + \boldsymbol{q}_{k,i}) \tag{10}$$

$$e_{k,L_k(i)-\Delta}^{(L_k(i))} = d_k(i) - \boldsymbol{u}_{k,i}(1:L_k(i)-\Delta)(\boldsymbol{\psi}_{k-1}^{(i)}(1:L_k(i)-\Delta) + \boldsymbol{q}_{k,i}(1:L_k(i)-\Delta)) \tag{11}$$

in which $\boldsymbol{u}_{k,i}(1:L_k(i)-\Delta)$, $\boldsymbol{\psi}_{k-1}^{(i)}(1:L_k(i)-\Delta)$ and $\boldsymbol{q}_{k,i}(1:L_k(i)-\Delta)$ consist of the initial $L_k(i)-\Delta$ entries of $\boldsymbol{u}_{k,i}$, $\boldsymbol{\psi}_{k-1}^{(i)}$ and $\boldsymbol{q}_{k,i}$ respectively. The local estimate of integer length $L_k(i)$ in node *k* is obtained by (5).

IV. PERFORMANCE ANALYSIS OF INCREMENTAL FT ALGORITHM IN THE PRESENCE OF NOISY LINKS

In this section, we derive a mathematical formulation for the steady-state tap-length of the incremental FT algorithm in the presence of noisy links. We consider a linear measurement model as:

$$d_k(i) = \boldsymbol{u}_{L_{opt}k,i} \boldsymbol{w}_{L_{opt}}^o + v_k(i) \tag{12}$$

where $v_k(i)$ is some spatially and temporally white noise with zero mean and variance $\sigma_{v,k}^2$ and independent of $\boldsymbol{u}_{L_{opt}\ell,j}$ and $d_\ell(j)$ for all ℓ and *j*. In (12), $\boldsymbol{u}_{L_{opt}k,i}$ is a vector of length L_{opt}. Since the length of the vectors changes iteration by iteration in this scenario, it is helpful to consider an upper bound for the length and make the length of the vectors equal to this upper bound by zero-padding them. So, we consider an upper bound for tap-length as:

$$L_{ub} > max\left\{L_{opt}\ , L_k(i)\right\} \qquad for\ all\quad i \tag{13}$$

By this definition, the unknown parameter $\boldsymbol{w}_{L_{opt}}^o$ is denoted as $\boldsymbol{w}_{L_{ub}}$, where $\boldsymbol{w}_{L_{ub}}$ is obtained by padding $\boldsymbol{w}_{L_{opt}}^o$ with $L_{ub}-L_{opt}$ zeros. Also, we define the vector $\boldsymbol{\psi}_{L_{ub},k-1}^{(i)}$ that is obtained by padding $\boldsymbol{\psi}_{k-1}^{(i)}$ with $L_{ub}-L_k(i)$ zeros. We partition the unknown vector $\boldsymbol{w}_{L_{ub}}$ as:

$$\begin{bmatrix}\dot{\boldsymbol{w}}\\ \ddot{\boldsymbol{w}}\\ \dddot{\boldsymbol{w}}\end{bmatrix} \tag{14}$$

where $\dot{\boldsymbol{w}}$ and $\ddot{\boldsymbol{w}}$ are the portions modeled by $\dot{\boldsymbol{\psi}}_{k-1}^{(i)}$ and $\ddot{\boldsymbol{\psi}}_{k-1}^{(i)}$**,** respectively, such that

$$\dot{\boldsymbol{\psi}}_{k-1}^{(i)} = \boldsymbol{\psi}_{k-1}^{(i)}(1:L_k(i)-\Delta) \tag{15}$$

$$\ddot{\boldsymbol{\psi}}_{k-1}^{(i)} = \boldsymbol{\psi}_{k-1}^{(i)}\left(L_k(i)-\Delta+1:L_k(i)\right) \tag{16}$$

where $\boldsymbol{\psi}_{k-1}^{(i)}(1:L_k(i)-\Delta)$ consists of the initial $L_k(i)-\Delta$ entries of $\boldsymbol{\psi}_{k-1}^{(i)}$ and $\boldsymbol{\psi}_{k-1}^{(i)}\left(L_k(i)-\Delta+1:L_k(i)\right)$ consists of the last Δ entries of $\boldsymbol{\psi}_{k-1}^{(i)}$ and $\dddot{\boldsymbol{w}}$ is the undermodeled part of $\boldsymbol{w}_{L_{ub}}$. We define $\overline{\boldsymbol{\psi}}_{\mathbf{L_{ub},k-1}}^{(\mathbf{i})} = \boldsymbol{w}_{\mathbf{L_{ub}}} - \boldsymbol{\psi}_{\mathbf{L_{ub},k-1}}^{(\mathbf{i})}$, which measures the difference between the desired parameter $\boldsymbol{w}_{L_{ub}}$ and its estimate at node $k-1$, and we partition it into three parts:

$$\overline{\boldsymbol{\psi}}_{\mathbf{L_{ub},k-1}}^{(\mathbf{i})} = \begin{bmatrix}\dot{\boldsymbol{w}}\\ \ddot{\boldsymbol{w}}\\ \dddot{\boldsymbol{w}}\end{bmatrix} - \begin{bmatrix}\dot{\boldsymbol{\psi}}_{k-1}^{(i)}\\ \ddot{\boldsymbol{\psi}}_{k-1}^{(i)}\\ \mathbf{0}_{(L_{ub}-L_k(i))\times 1}\end{bmatrix} = \begin{bmatrix}\dot{\overline{\boldsymbol{\psi}}}_{k-1}^{(i)}\\ \ddot{\overline{\boldsymbol{\psi}}}_{k-1}^{(i)}\\ \dddot{\overline{\boldsymbol{\psi}}}_{k-1}^{(i)}\end{bmatrix} \tag{17}$$

For the ease of analysis, also, we partition the regression vector $\boldsymbol{u}_{L_{ub},k,i}$ with length L_{ub} into three parts $\dot{\boldsymbol{u}}_{k,i}$, $\ddot{\boldsymbol{u}}_{k,i}$ and $\dddot{\boldsymbol{u}}_{k,i}$. Also, we partition the channel noise vector $\boldsymbol{q}_{k,i}$ into two parts $\dot{\boldsymbol{q}}_{k,i}$ and $\ddot{\boldsymbol{q}}_{k,i}$ corresponding to $\dot{\boldsymbol{\psi}}_{k-1}^{(i)}$ and $\ddot{\boldsymbol{\psi}}_{k-1}^{(i)}$, respectively. With these definitions and substituting (12) into (10) and (11) and by zero-padding all of the vectors in (12), (10), and (11) to make their length equal to L_{ub}, we have:

$$e_{k,L_k(i)}^{(L_k(i))} = \boldsymbol{u}_{L_{opt}k,i}\boldsymbol{w}_{L_{opt}}^{o} - \boldsymbol{u}_{k,i}\boldsymbol{\psi}_{k-1}^{(i)} - \boldsymbol{u}_{k,i}\boldsymbol{q}_{k,i} + v_k(i) \tag{18}$$
$$= \boldsymbol{u}_{L_{ub},k,i}\begin{bmatrix}\boldsymbol{w}_{L_{opt}}^{\mathbf{o}} \\ \mathbf{0}_{(L_{ub}-L_{opt})\times 1}\end{bmatrix} - \boldsymbol{u}_{L_{ub},k,i}\begin{bmatrix}\boldsymbol{\psi}_{k-1}^{(i)} \\ \mathbf{0}_{(L_{ub}-L_k(i))\times 1}\end{bmatrix} - \boldsymbol{u}_{L_{ub},k,i}\begin{bmatrix}\boldsymbol{q}_{k,i} \\ \mathbf{0}_{(L_{ub}-L_k(i))\times 1}\end{bmatrix} + v_k(i)$$
$$= \boldsymbol{u}_{L_{ub},k,i}\bar{\boldsymbol{\psi}}_{L_{ub},k-1}^{(i)} - \boldsymbol{u}_{L_{ub},k,i}\begin{bmatrix}\boldsymbol{q}_{k,i} \\ \mathbf{0}_{(L_{ub}-L_k(i))\times 1}\end{bmatrix} + v_k(i)$$
$$= \begin{bmatrix}\dot{\boldsymbol{u}}_{k,i} & \ddot{\boldsymbol{u}}_{k,i} & \dddot{\boldsymbol{u}}_{k,i}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{\psi}}_{k-1}^{(i)} \\ \ddot{\boldsymbol{\psi}}_{k-1}^{(i)} \\ \dddot{\boldsymbol{w}}\end{bmatrix} - \begin{bmatrix}\dot{\boldsymbol{u}}_{k,i} & \ddot{\boldsymbol{u}}_{k,i} & \dddot{\boldsymbol{u}}_{k,i}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{q}}_{k,i} \\ \ddot{\boldsymbol{q}}_{k,i} \\ \mathbf{0}_{(L_{ub}-L_k(i))\times 1}\end{bmatrix} + v_k(i)$$
$$= \dot{\boldsymbol{u}}_{k,i}\dot{\boldsymbol{\psi}}_{k-1}^{(i)} + \ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{\psi}}_{k-1}^{(i)} + \dddot{\boldsymbol{u}}_{k,i}\dddot{\boldsymbol{w}} - \dot{\boldsymbol{u}}_{k,i}\dot{\boldsymbol{q}}_{k,i} - \ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{q}}_{k,i} + v_k(i)$$

and

$$e_{k,L_k(i)-\Delta}^{(L_k(i))} = \boldsymbol{u}_{L_{opt}k,i}\boldsymbol{w}_{L_{opt}}^{o} - \boldsymbol{u}_{k,i}(1:L_k(i)-\Delta)\boldsymbol{\psi}_{k-1}^{(i)}(1:L_k(i)-\Delta) \tag{19}$$
$$- \boldsymbol{u}_{k,i}(1:L_k(i)-\Delta)\boldsymbol{q}_{k,i}(1:L_k(i)-\Delta) + v_k(i)$$
$$= \boldsymbol{u}_{L_{ub}k,i}\begin{bmatrix}\dot{\boldsymbol{w}} - \dot{\boldsymbol{\psi}}_{k-1}^{(i)} \\ \ddot{\boldsymbol{w}} \\ \dddot{\boldsymbol{w}}\end{bmatrix} - \boldsymbol{u}_{L_{ub}k,i}\begin{bmatrix}\dot{\boldsymbol{q}}_{k,i} \\ \mathbf{0}_{(L_{ub}-L_k(i)+\Delta)\times 1}\end{bmatrix} + v_k(i)$$
$$= \dot{\boldsymbol{u}}_{k,i}\dot{\boldsymbol{\psi}}_{k-1}^{(i)} + \ddot{\boldsymbol{u}}_{k,i}\ddot{\mathbf{w}} + \dddot{\boldsymbol{u}}_{k,i}\dddot{\mathbf{w}} - \dot{\boldsymbol{u}}_{k,i}\dot{\boldsymbol{q}}_{k,i} + v_k(i)$$

Substituting (18) and (19) in the key term in (9) results:

$$\left(e_{k,L_k(i)}^{(L_k(i))}\right)^2 - \left(e_{k,L_k(i)-\Delta}^{(L_k(i))}\right)^2 \tag{20}$$
$$= A - B + C - D + E - F + G - H$$
$$- I - J - K + L - M + N - O + P$$

where

$$A = 2v_k(i)\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{\psi}}_{k-1}^{(i)},\ B = 2v_k(i)\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{w}} \tag{21}$$
$$C = 2\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{\psi}}_{k-1}^{(i)}\dot{\boldsymbol{u}}_{k,i}\dot{\boldsymbol{\psi}}_{k-1}^{(i)},\ D = 2\dot{\boldsymbol{u}}_{k,i}\dot{\boldsymbol{\psi}}_{k-1}^{(i)}\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{w}}$$
$$E = \left[\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{\psi}}_{k-1}^{(i)}\right]^2,\ F = \left[\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{w}}\right]^2$$
$$G = 2\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{w}}\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{\psi}}_{k-1}^{(i)},\ H = 2\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{w}}\ \ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{w}}$$
$$I = 2v_k(i)\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{q}}_{k,i},\ J = 2\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{\psi}}_{k-1}^{(i)}\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{q}}_{k,i}$$
$$K = 2\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{\psi}}_{k-1}^{(i)}\dot{\boldsymbol{u}}_{k,i}\dot{\boldsymbol{q}}_{k,i},\ L = 2\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{w}}\dot{\boldsymbol{u}}_{k,i}\dot{\boldsymbol{q}}_{k,i}$$
$$M = 2\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{q}}_{k,i}\dot{\boldsymbol{u}}_{k,i}\dot{\boldsymbol{\psi}}_{k-1}^{(i)},\ N = \left[\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{q}}_{k,i}\right]^2$$
$$O = 2\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{q}}_{k,i}\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{w}},\ P = 2\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{q}}_{k,i}\dot{\boldsymbol{u}}_{k,i}\dot{\boldsymbol{q}}_{k,i}$$

From (9), and (20) we have:

$$\ell_{k,f}(i) = \left(\ell_{k-1,f}(i) + \ell_{q,k,i}\right) - a_k \\ -\gamma_k\left[A - B + C - D + E - F + G - H - I - J - K + L - M + N - O + P\right] \tag{22}$$

Now, based on (22), we perform the steady-state analysis. The following assumptions, commonly considered in the literature of adaptive networks, are made to simplify this analysis.

1) The regressors $\boldsymbol{u}_{L_{ub}k,i}$ are temporally and spatially independent and the components of them are drawn from a zero-mean white Gaussian process with variance $\sigma_{u,k}^2$.
2) The channel noise $\boldsymbol{q}_{k,i}$ is independent of $\boldsymbol{u}_{L_{ub}\ell,j}$, $\boldsymbol{q}_{\ell,j}$ and $v_\ell(j)$ for all ℓ, j;
3) The tap-length at each sensor will converge to a fixed value $L_k(\infty)$ at steady-state.
4) The tail entries of the unknown parameter $\boldsymbol{w}_{L_{opt}}^o$ are assumed to be drawn from a zero-mean white signal with variance σ_o^2.
5) In the steady-state $\left\{\begin{bmatrix}\dot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)} \\ \ddot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)}\end{bmatrix}\begin{bmatrix}\dot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)} \\ \ddot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)}\end{bmatrix}^T\right\} = \sigma_{\bar{\psi},k-1}^2 I$, where $\sigma_{\bar{\psi},k-1}^2$ is the variance of the elements of $\dot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)}$ and $\ddot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)}$.

Assigning $L_k(\infty)$ and $L_{k-1}(\infty)$ as the steady-state tap-length of nodes k and $k-1$, and considering that $E\left\{\ell_{q,k,i}\right\} = 0$, we take expectations from both sides of (22) in the steady-state:

$$L_k(\infty) = \left(L_{k-1}(\infty) - a_k\right) \\ -\gamma_k E\left\{A - B + C - D + E - F + G - H - I - J - K + L - M + N - O + P\right\}, i \to \infty \tag{23}$$

Using the assumptions (1)-(5), the moments in (23) can be calculated as:

$$E\{A\} = E\{B\} = E\{C\} = E\{D\} = E\{G\} = E\{H\} = E\{I\} \\ = E\{J\} = E\{K\} = E\{L\} = E\{M\} = E\{O\} = E\{P\} = 0 \tag{24}$$

$$E\{E\} = E\left\{\left[\ddot{\boldsymbol{u}}_{k,i}\ddot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)}\right]^2\right\} = E\left(tr\left\{\ddot{\boldsymbol{u}}_{k,i}^T\ddot{\boldsymbol{u}}_{k,i}\ddot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)}\ddot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)\,T}\right\}\right) \\ = tr\left(\sigma_{u,k}^2 I_\Delta E\left\{\ddot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)}\ddot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)\,T}\right\}\right) = \sigma_{u,k}^2 E\left\{\left\|\ddot{\bar{\boldsymbol{\psi}}}_{k-1}^{(i)}\right\|^2\right\} \tag{25}$$

$$E\{F\} = E\left\{\left[\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{w}}\right]^2\right\} = \sigma_{u,k}^2 E\left\{\left\|\ddot{\boldsymbol{w}}\right\|^2\right\} \tag{26}$$

$$E\{N\} = E\left\{\left[\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{q}}_{k,i}\right]^2\right\} = E\left(tr\left\{\ddot{\boldsymbol{u}}_{k,i}^T\ddot{\boldsymbol{u}}_{k,i}\ddot{\boldsymbol{q}}_{k,i}\ddot{\boldsymbol{q}}_{k,i}^T\right\}\right) \\ = tr\left(E\left\{\ddot{\boldsymbol{u}}_{k,i}^T\ddot{\boldsymbol{u}}_{k,i}\right\}E\left\{\ddot{\boldsymbol{q}}_{k,i}\ddot{\boldsymbol{q}}_{k,i}^T\right\}\right) = \sigma_{u,k}^2 tr\left(E\left\{\ddot{\boldsymbol{q}}_{k,i}\ddot{\boldsymbol{q}}_{k,i}^T\right\}\right) = \sigma_{u,k}^2 tr\left(\ddot{\boldsymbol{Q}}_k\right) \tag{27}$$

In (27), $\ddot{\boldsymbol{Q}}_k$ is a matrix that includes only the last Δ rows and columns of $\boldsymbol{Q}_k$. Substituting (24)-

(27) in (23) results:

$$L_k(\infty) = \left(L_{k-1}(\infty) - a_k\right) - \gamma_k \sigma_{u,k}^2 \left\{ E\left\{ \left\| \dddot{\boldsymbol{\psi}}_{k-1}^{(i)} \right\|^2 \right\} - E\left\{ \left\| \dddot{\mathbf{w}} \right\|^2 \right\} + tr\left(\ddot{\boldsymbol{Q}}_k \right) \right\},\ i \to \infty \tag{28}$$

By defining $\boldsymbol{P}_{L_{ub},k-1} = \overline{\boldsymbol{\psi}}_{L_{ub},k-1}^{(\infty)}$ and partitioning $\boldsymbol{P}_{L_{ub},k-1}$ similar to $\overline{\boldsymbol{\psi}}_{L_{ub},k-1}^{(i)}$ in (17) into three parts $\dot{\boldsymbol{P}}_{k-1}$, $\ddot{\boldsymbol{P}}_{k-1}$ and $\dddot{\boldsymbol{P}}_{k-1}$, we can rewrite (28) as:

$$L_k(\infty) = \left(L_{k-1}(\infty) - a_k\right) - \gamma_k \sigma_{u,k}^2 \left\{ E\left\{ \left\| \dddot{\boldsymbol{P}}_{k-1} \right\|^2 \right\} - E\left\{ \left\| \dddot{\mathbf{w}} \right\|^2 \right\} + tr\left(\ddot{\boldsymbol{Q}}_k \right) \right\} \tag{29}$$

To evaluate the term $E\left\{ \left\| \dddot{\boldsymbol{P}}_{k-1} \right\|^2 \right\}$, we use the reported results in [24]. According to the performed analysis in [24]:

$$E\left\{ \left\| \boldsymbol{P}_{L_{ub},k-1} \right\|^2 \right\} = \left(1 - \Pi_{k,1}\right)^{-1} s_k \tag{30}$$

where

$$\Pi_{k,\ell} \triangleq \beta_{k-1}\beta_{k-2}\ldots\beta_1\beta_N\beta_{N-1}\ldots\beta_{k+\ell}\,\beta_{k+\ell-1} \tag{31}$$

$$s_k \triangleq \Pi_{k,2}\, f_k + \Pi_{k,3}\, f_{k+1} + \ldots + \Pi_{k,N-1}\, f_{k-3} + \Pi_{k,N}\, f_{k-2} + f_{k-1} \tag{32}$$

with

$$\beta_k = 1 - 2\mu_k \sigma_{u,k}^2 + \mu_k^2 \sigma_{u,k}^4 \left(L_{k-1}(\infty) + 2 \right) \tag{33}$$

$$f_k = 2\mu_k \sigma_{u,k}^2 \left(1 - \mu_k \sigma_{u,k}^2\right) \left\| \dddot{\mathbf{w}} \right\|^2 + \mu_k^2 \sigma_{v,k}^2 \sigma_{u,k}^2 L_{k-1}(\infty) + \beta_k tr\left(\mathbf{Q}_k \right) \tag{34}$$

Now, using (30)-(34), we compute $E\left\{ \left\| \dddot{\boldsymbol{P}}_{k-1} \right\|^2 \right\}$. To do so; we rewrite the steady-state mean squared deviation (MSD) as follows:

$$E\left\{ \left\| \boldsymbol{P}_{L_{ub},k-1} \right\|^2 \right\} = E\left\{ \left\| \dot{\boldsymbol{P}}_{k-1} \right\|^2 \right\} + E\left\{ \left\| \ddot{\boldsymbol{P}}_{k-1} \right\|^2 \right\} + E\left\{ \left\| \dddot{\mathbf{w}} \right\|^2 \right\} \tag{35}$$

According to assumption (5), we have:

$$E\left\{ \left\| \dot{\boldsymbol{P}}_{k-1} \right\|^2 \right\} = \left(L_{k-1}(\infty) - \Delta \right) \sigma_{\overline{\psi},k-1}^2 \tag{36}$$

$$E\left\{ \left\| \ddot{\boldsymbol{P}}_{k-1} \right\|^2 \right\} = \Delta \sigma_{\overline{\psi},k-1}^2 \tag{37}$$

and so

$$E\left\{ \left\| \boldsymbol{P}_{L_{ub},k-1} \right\|^2 \right\} = L_{k-1}(\infty) \sigma_{\overline{\psi},k-1}^2 + E\left\{ \left\| \dddot{\mathbf{w}} \right\|^2 \right\} \tag{38}$$

From (37) and (38), we have:

$$E\left\{\left\|\boldsymbol{P}_{L_{ub},k-1}\right\|^2\right\}=\frac{L_{k-1}(\infty)}{\Delta}E\left\{\left\|\ddot{\boldsymbol{P}}_{k-1}\right\|^2\right\}+E\left\{\left\|\dddot{\boldsymbol{w}}\right\|^2\right\} \tag{39}$$

According to (30) and (39) we have:

$$E\left\{\left\|\ddot{\boldsymbol{P}}_{k-1}\right\|^2\right\}=\frac{\Delta}{L_{k-1}(\infty)}\left(1-\Pi_{k,1}\right)^{-1}s_k-\frac{\Delta}{L_{k-1}(\infty)}E\left\{\left\|\dddot{\boldsymbol{w}}\right\|^2\right\} \tag{40}$$

Substituting (40) into (29) results:

$$\begin{aligned}L_k(\infty)&=\left(L_{k-1}(\infty)-a_k\right)\\&\quad-\gamma_k\sigma_{u,k}^2\left\{\frac{\Delta}{L_{k-1}(\infty)}\left(1-\Pi_{k,1}\right)^{-1}s_k-\frac{\Delta}{L_{k-1}(\infty)}E\left\{\left\|\ddot{\boldsymbol{w}}\right\|^2\right\}-E\left\{\left\|\dddot{\boldsymbol{w}}\right\|^2\right\}+tr\left(\ddot{\boldsymbol{Q}}_k\right)\right\}\end{aligned} \tag{41}$$

The values of moments $E\left\{\left\|\ddot{\boldsymbol{w}}\right\|^2\right\}$ and $E\left\{\left\|\dddot{\boldsymbol{w}}\right\|^2\right\}$ in (41) depend on the relative values of $L_{k-1}(\infty)$ and L_{opt}. If $\left(L_{k-1}(\infty)\text{-}\Delta\right)>L_{opt}$, then in the steady-state $\ddot{\boldsymbol{w}}=0$, but, if $\left(L_{k-1}(\infty)\text{-}\Delta\right)\le L_{opt}$, then for $L_{k-1}(\infty)>L_{opt}$, $\ddot{\boldsymbol{w}}$ consists of $L_{opt}\text{-}L_{k-1}(\infty)+\Delta$ nonzero elements of $\boldsymbol{w}_{L_{opt}}^{\mathbf{o}}$ and for $L_{k-1}(\infty)\le L_{opt}$ it consists of Δ nonzero elements of $\boldsymbol{w}_{L_{opt}}^{\mathbf{o}}$, so we have:

$$E\left\{\left\|\ddot{\boldsymbol{w}}\right\|^2\right\}=\begin{cases}0 & if\ \ L_{k-1}(\infty)>L_{opt}+\Delta\\ \left(L_{opt}\text{-}L_{k-1}(\infty)+\Delta\right)\sigma_o^2, & if\ \ L_{opt}<L_{k-1}(\infty)\le L_{opt}+\Delta\\ \Delta\sigma_o^2 & if\ L_{k-1}(\infty)\le L_{opt}\ and\ L_{k-1}(\infty)\le L_{opt}+\Delta\end{cases} \tag{42}$$

About $\dddot{\boldsymbol{w}}$, if $L_{k-1}(\infty)>L_{opt}$ then in the steady-state $\dddot{\boldsymbol{w}}=\boldsymbol{0}$, but, if $L_{k-1}(\infty)\le L_{opt}$ then $\dddot{\boldsymbol{w}}$ consists of $L_{opt}\text{-}L_{k-1}(\infty)$ nonzero elements of $\boldsymbol{w}_{L_{opt}}^{\mathbf{o}}$, so we have:

$$E\left\{\left\|\dddot{\boldsymbol{w}}\right\|^2\right\}=\begin{cases}0 & if\ \ L_{k-1}(\infty)>L_{opt}\\ \left(L_{opt}\text{-}L_{k-1}(\infty)\right)\sigma_o^2, & if\ \ L_{k-1}(\infty)\le L_{opt}\end{cases} \tag{43}$$

In fact, even if $L_k(\infty)$ and $L_{k-1}(\infty)$ are not precisely equal, they will be approximately equal, then we temporarily assume $L_k(\infty)=L_{k-1}(\infty)$, and rewrite (41) as:

$$\frac{\Delta}{L_{k-1}(\infty)}\left(1-\Pi_{k,1}\right)^{-1}s_k-\frac{\Delta}{L_{k-1}(\infty)}E\left\{\left\|\ddot{\boldsymbol{w}}\right\|^2\right\}-E\left\{\left\|\dddot{\boldsymbol{w}}\right\|^2\right\}+tr\left(\ddot{\boldsymbol{Q}}_k\right)=-\frac{a_k}{\gamma_k\sigma_{u,k}^2} \tag{44}$$

From (42) and (43), it is observed that for $L_{k-1}(\infty)>\left(L_{opt}+\Delta\right)$, both $E\left\{\left\|\ddot{\boldsymbol{w}}\right\|^2\right\}$ and $E\left\{\left\|\dddot{\boldsymbol{w}}\right\|^2\right\}$ are zero. So, the left-hand side of (44) is non-negative, but the right-hand side is negative. Hence, in the presence of noisy links, indeed $L_{k-1}(\infty)\le\left(L_{opt}+\Delta\right)$, in other words, the error caused by the noisy links will not lead to the overestimation of tap-length.

Our studies have shown that under high noise conditions, there will be no convergence for length. In this condition, the tap-length will fluctuate sharply and will not tend to a constant value. However, under low noise conditions, as we will show, the length can still be slightly larger than the optimal tap-length. According to these results, we have:

$$if \quad L_{opt} < L_{k-1}(\infty) \le L_{opt} + \Delta \Rightarrow \begin{cases} E\left\{\|\ddot{\boldsymbol{w}}\|^2\right\} = \left(L_{opt} - L_{k-1}(\infty) + \Delta\right)\sigma_o^2 \\ E\left\{\|\ddot{\boldsymbol{w}}\|^2\right\} = 0 \end{cases} \tag{45}$$

So, we can rewrite $\frac{\Delta}{L_{k-1}(\infty)} E\left\{\|\ddot{\boldsymbol{w}}\|^2\right\} + E\left\{\|\ddot{\boldsymbol{w}}\|^2\right\}$ in (41) as:

$$\frac{\Delta}{L_{k-1}(\infty)} E\left\{\|\ddot{\boldsymbol{w}}\|^2\right\} + E\left\{\|\ddot{\boldsymbol{w}}\|^2\right\} = \left(L_{opt} - L_{k-1}(\infty) + \Delta\right)\sigma_o^2 \tag{46}$$

Substituting (46) into (41) results:

$$\begin{aligned} L_k(\infty) &= \left(L_{k-1}(\infty) - a_k\right) \\ &\quad - \gamma_k \sigma_{u,k}^2 \left\{ \frac{\Delta}{L_{k-1}(\infty)} \left(1 - \Pi_{k,1}\right)^{-1} s_k - \left(L_{opt} - L_{k-1}(\infty) + \Delta\right)\sigma_o^2 + tr\left(\ddot{\boldsymbol{Q}}_k\right) \right\} \end{aligned} \tag{47}$$

Equation (47) can be arranged in the following manner:

$$\begin{aligned} L_k(\infty) &= \left(1 - \gamma_k \sigma_{u,k}^2 \, \sigma_o^2\right) L_{k-1}(\infty) \\ &\quad + \gamma_k \sigma_{u,k}^2 \sigma_o^2 \times \left\{ L_{opt} + \Delta - \frac{\Delta}{\sigma_o^2} \left(1 - \Pi_{k,1}\right)^{-1} \dot{s}_k - \frac{1}{\sigma_o^2} tr\left(\ddot{\boldsymbol{Q}}_k\right) \right\} - a_k \end{aligned} \tag{48}$$

where

$$\dot{s}_k \triangleq \frac{s_k}{L_{k-1}(\infty)} = \Pi_{k,2}\, \dot{f}_k + \Pi_{k,3}\, \dot{f}_{k+1} + \ldots + \Pi_{k,N-1}\, \dot{f}_{k-3} + \Pi_{k,N}\, \dot{f}_{k-2} + \dot{f}_{k-1} \tag{49}$$

and

$$\dot{f}_k = \frac{f_k}{L_{k-1}(\infty)} = \mu_k^2 \sigma_{v,k}^2 \sigma_{u,k}^2 + \dot{\beta}_k tr\left(\boldsymbol{Q}_k\right) \tag{50}$$

where the first term of f_k is neglected, since $\ddot{\boldsymbol{w}}$ is zero. The term $\dot{\beta}_k$ is defined as:

$$\dot{\beta}_k = \frac{\beta_k}{L_{k-1}(\infty)} = \left(1 - 2\mu_k \sigma_{u,k}^2 + \mu_k^2 \sigma_{u,k}^4 \left(L_{opt} + 2\right)\right) / L_{opt} \tag{51}$$

where, for the sake of simplicity $L_{k-1}(\infty)$ is approximated by L_{opt}. If the step size is chosen considerably small, then the term $E\left\{\left\|\ddot{\boldsymbol{P}}_{k-1}\right\|^2\right\}$ will be small enough. Therefore, the applied approximations in $\dot{f}_k$ and $\dot{\beta}_k$ will not considerably affect the steady-state tap-length value. By defining h_k and g_k as:

$$h_k = 1\text{-}\gamma_k \sigma_{u,k}^2 \sigma_o^2 \tag{52}$$

$$g_k = \gamma_k \sigma_{u,k}^2 \sigma_o^2 \left\{ L_{opt} + \Delta\text{-}\frac{\Delta}{\sigma_o^2}\left(1-\Pi_{k,1}\right)^{-1} \dot{s}_k - \frac{1}{\sigma_o^2} tr\left(\ddot{\mathbf{Q}}_k\right) \right\} - a_k \tag{53}$$

equation (48) can be written as:

$$L_k(\infty) = h_k L_{k-1}(\infty) + g_k \tag{54}$$

This equation involves both $L_k(\infty)$ and $L_{k-1}(\infty)$, i.e., data from two spatial locations. To solve this challenge, we use ring topology [2]-[4]. In this manner, we will have:

$$L_{k-1}(\infty) = \left(1-\Gamma_{k,1}\right)^{-1} m_k \tag{55}$$

where

$$\Gamma_{k,\ell} \triangleq h_{k-1} h_{k-2} \dots h_1 h_N h_{N-1} \dots h_{k+\ell} h_{k+\ell-1}, \quad \ell = 1, \dots, N \tag{56}$$

$$m_k \triangleq \Gamma_{k,2} g_k + \Gamma_{k,3} g_{k+1} + \dots + \Gamma_{k,N-1} g_{k-3} + \Gamma_{k,N} g_{k-2} + g_{k-1} \tag{57}$$

Due to the complicated form of (55), it is not easy to derive straightforward interpretations about the steady-state tap-length. To obtain a clear insight, we simplify this equation. First, we assume a same tap-length adaption step size for all nodes, i.e., $\gamma_k = \gamma, \forall k \in \mathrm{N}$, also we assume that $\mathbf{R}_{u,k} = \sigma_u^2 \mathbf{I}$. With these assumptions, we have:

$$h_k = 1\text{-}\gamma \sigma_u^2 \sigma_o^2 \tag{58}$$

Furthermore, we assume $\gamma\sigma_u^2\sigma_o^2 \ll 1$ such that:

$$\Gamma_{k,1} = h_1 h_2 \dots h_N = \left(1\text{-}\gamma\sigma_u^2\sigma_o^2\right)^N \approx 1\text{-}N\gamma\sigma_u^2\sigma_o^2 \tag{59}$$

Also, m_k can be approximated as:

$$m_k \approx \sum_{k=1}^{N} g_k = N\gamma\sigma_u^2\sigma_o^2\left(L_{opt}+\Delta\right) - \gamma\sigma_u^2\Delta\sum_{k=1}^{N}\left(1-\Pi_{k,1}\right)^{-1}\dot{s}_k - \gamma\sigma_u^2\sum_{k=1}^{N} tr\left(\ddot{\mathbf{Q}}_k\right) - \sum_{k=1}^{N} a_k \tag{60}$$

Substituting (59) and (60) in (55) results:

$$L_{k-1}(\infty) = \left(L_{opt}+\Delta\right) - \frac{\Delta}{N\sigma_o^2}\sum_{k=1}^{N}\left(1-\Pi_{k,1}\right)^{-1}\dot{s}_k - \frac{1}{N\sigma_o^2}\sum_{k=1}^{N} tr\left(\ddot{\mathbf{Q}}_k\right) - \frac{\sum_{k=1}^{N} a_k}{N\gamma\sigma_u^2\sigma_o^2} \tag{61}$$

From (61), several results can be concluded as follows:

1) In the steady-state tap-length relation, there isn't any trace of the noise $\ell_{q,k,i}$ added to the local estimate of the fractional tap-length, and the noise term added to the local estimation of the unknown parameter, $\boldsymbol{q}_{k,i}$, only appears in it. So, $\ell_{q,k,i}$ doesn't have any effect on the steady-state tap-length.

This is perhaps unexpected, but it can be justified by noting that in (9), $\ell_{q,k,i}$ plays the same role as the leakage factor. In other words, proper selection of the leakage factor could compensate for the adverse effect of $\ell_{q,k,i}$.

2) The noise appears in two terms $\frac{\Delta}{\sigma_o^2}\left(1-\Pi_{k,1}\right)^{-1}\dot{s}_k$ and $\frac{1}{\sigma_o^2}tr\left(\ddot{\boldsymbol{Q}}_k\right)$, where their difference affects the steady-state tap-length relationship (61). As mentioned earlier, as the noise level increases, the steady-state tap-length decreases compared to the ideal link version. But, under low noise conditions, this length is still larger than the optimal filter length.

V. SIMULATIONS

To confirm the derived theoretical equations, we provide numerical simulation results in this section. The steady-state tables and curves are resulted from averaging the last 1000 samples of 5,000 iterations. All results are averaged over 100 independent runs of the same experiment. We consider a network with *N=12* sensors investigating a desired vector with length L_{opt}. According to assumption (4), the unknown parameter elements are drawn from a random white signal with zero mean and variance $\sigma_o^2 = 0.1$.

The regressors and observation noise are independent zero-mean Gaussian with $\sigma_{u,k}^2 \in [0.5, 2]$ and $\sigma_{v,k}^2 \in [0.01, 0.1]$ for every node *k*. The step size for the DILMS algorithm is set as $\mu_k = 0.01$ for all sensors. For the incremental FT algorithm, the parameters are chosen as $\delta_k = 1$, and $\Delta = 4$. The initial length is set equal to the minimum tap-length $L_{min} = 6$.

Table I shows the simulated steady-state tap-lengths, together with the theoretical results from expression (55) for $L_{opt} = 40$ and channel noise with statistics $\boldsymbol{Q}_k = 10^{-5}\mathbf{I}, \sigma_{c,k}^2 = 10^{-5}, \boldsymbol{Q}_k = 10^{-4}\mathbf{I}, \sigma_{c,k}^2 = 10^{-4}$ and $\boldsymbol{Q}_k = 10^{-3}\mathbf{I}, \sigma_{c,k}^2 = 10^{-3}$.

Since with increasing noise, the tap-length fluctuates sharply, so by increasing the noise level, we reduced the parameters γ_k and a_k to decrease the fluctuations. On this basis, we set γ_k and a_k as $(\gamma_k = 1, a_k = 0.001), (\gamma_k = 0.5, a_k = 0.001)$, and $(\gamma_k = 0.1, a_k = 0.00001)$ for noise levels $\boldsymbol{Q}_k = 10^{-5}\mathbf{I}, \sigma_{c,k}^2 = 10^{-5}, \boldsymbol{Q}_k = 10^{-4}\mathbf{I}, \sigma_{c,k}^2 = 10^{-4}$ and $\boldsymbol{Q}_k = 10^{-3}\mathbf{I}, \sigma_{c,k}^2 = 10^{-3}$ respectively. As this table shows, there is a good match between the simulations and the theoretical results.

Table I. Simulated and theoretical steady-state tap-lengths for different nodes, with $L_{opt} = 40$.

Node	$\boldsymbol{Q}_{\mathbf{k}} = \mathbf{10}^{-5}\mathbf{I},\ \sigma^2_{\mathbf{c,k}} = \mathbf{10}^{-5}$		$\boldsymbol{Q}_{\mathbf{k}} = \mathbf{10}^{-4}\mathbf{I},\ \sigma^2_{\mathbf{c,k}} = \mathbf{10}^{-4}$		$\boldsymbol{Q}_{\mathbf{k}} = \mathbf{10}^{-3}\mathbf{I},\ \sigma^2_{\mathbf{c,k}} = \mathbf{10}^{-3}$	
	Theory tap-length	***Simulated tap-length***	***Theory tap-length***	***Simulated tap-length***	***Theory tap-length***	***Simulated tap-length***
1	43.9517	43.9004	43.7384	43.4561	41.7006	41.4285
2	43.9514	43.8999	43.7381	43.4553	41.7007	41.4292
3	43.9510	43.8993	43.7377	43.4552	41.7006	41.4305
4	43.9512	43.8979	43.7379	43.4565	41.7006	41.4334
5	43.9512	43.8984	43.7379	43.4574	41.7007	41.4307
6	43.9506	43.8980	43.7373	43.4564	41.7006	41.4308
7	43.9509	43.8964	43.7375	43.4568	41.7004	41.4289
8	43.9514	43.8973	43.7379	43.4562	41.7005	41.4285
9	43.9514	43.8962	43.7381	43.4550	41.7005	41.4293
10	43.9509	43.8971	43.7376	43.4547	41.7005	41.4290
11	43.9510	43.8967	43.7376	43.4551	41.7004	41.4298
12	43.9511	43.8971	43.7376	43.4562	41.7004	41.4301

Also, as the theoretical findings show, the steady-state tap length decreases with the increasing noise level. However, as long as the noise level is low, the length reaches a fixed value.

The evolution curves of the fractional tap-length with the derived theoretical steady-state tap-length for node $k = 1$ and channel noise with different statistics are shown in Fig. 1 to provide clear insight. It can be observed from this figure that the presence of noisy links decreases the steady-state tap-length compared to the ideal link version, but as long as the noise level is low, this does not cause the under-modeling phenomenon. Also, this figure illustrates that the theoretical results match well with the simulations.

In the next simulation, the tap-length is considered to be $L_{opt} = 20$. The setup for this simulation is the same as those in the previous simulation.

Table II shows the simulated steady-state tap-lengths, together with the theoretical results for all nodes and channel noise with statistics $\boldsymbol{Q}_k = 10^{-5}\mathbf{I}, \sigma^2_{c,k} = 10^{-5}, \boldsymbol{Q}_k = 10^{-4}\mathbf{I}, \sigma^2_{c,k} = 10^{-4}$ and $\boldsymbol{Q}_k = 10^{-3}\mathbf{I}, \sigma^2_{c,k} = 10^{-3}$.

Fig. 2 shows the evolution curve of the fractional tap-length with the derived theoretical steady-state tap-length for node $k = 1$. Obviously, the derived theoretical expression for tap-length can predict the steady-state performance of the incremental FT algorithm in the presence of noisy links.

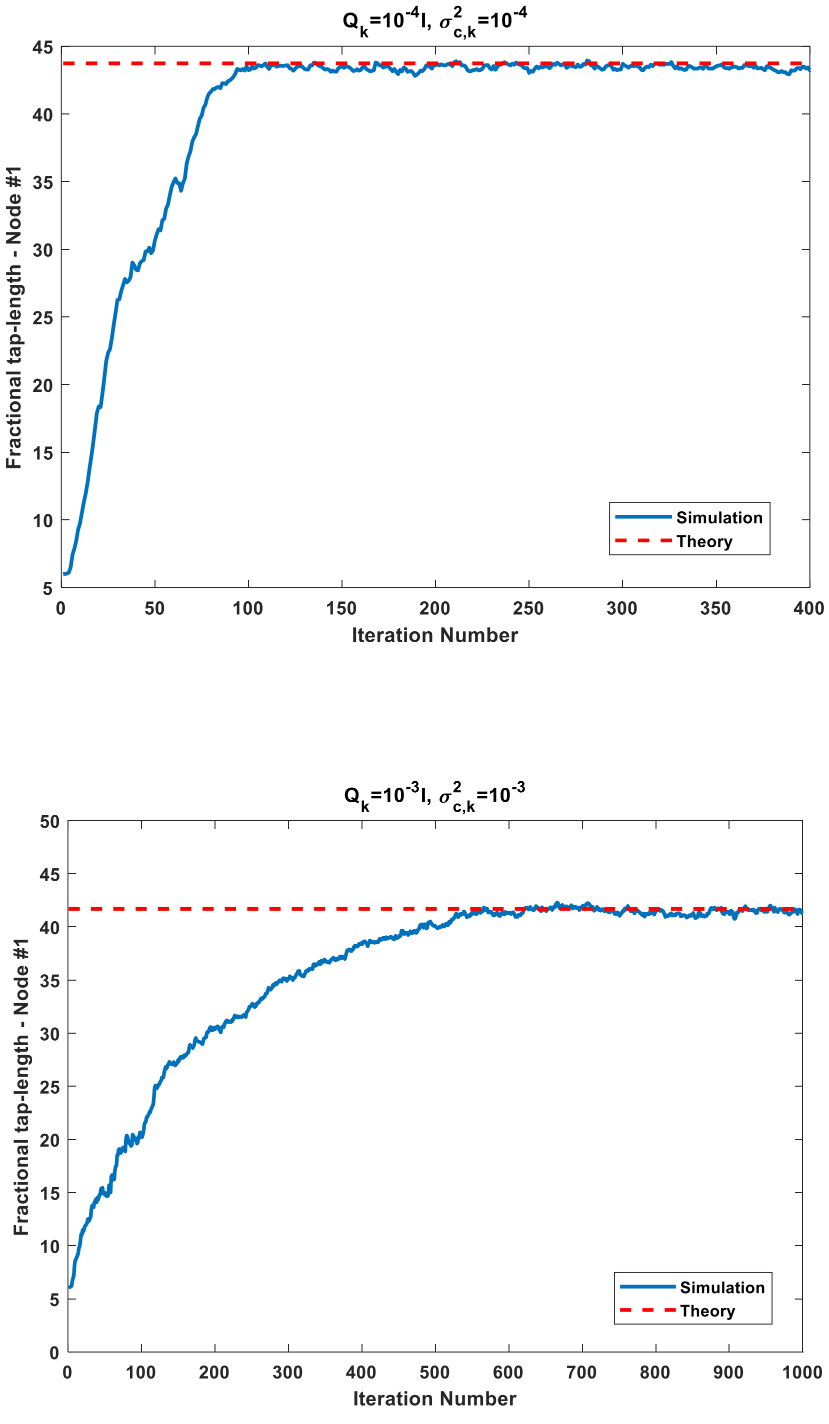

Fig. 1. Evolution curves of the fractional tap-length with the derived theoretical steady-state tap-length for node $k=1$, and for $L_{opt}=40$.

Table II. Simulated and theoretical steady-state tap-lengths for different nodes, with $L_{opt} = 20$.

Node	$\mathbf{Q}_k = 10^{-5}\mathbf{I},\ \sigma^2_{c,k} = 10^{-5}$		$\mathbf{Q}_k = 10^{-4}\mathbf{I},\ \sigma^2_{c,k} = 10^{-4}$		$\mathbf{Q}_k = 10^{-3}\mathbf{I},\ \sigma^2_{c,k} = 10^{-3}$	
	Theory tap-length	***Simulated tap-length***	***Theory tap-length***	***Simulated tap-length***	***Theory tap-length***	***Simulated tap-length***
1	23.9584	24.1352	23.7787	23.8926	22.0775	22.5759
2	23.9581	24.1366	23.7785	23.8920	22.0775	22.5764
3	23.9577	24.1365	23.7781	23.8917	22.0775	22.5764
4	23.9579	24.1369	23.7783	23.8928	22.0775	22.5781
5	23.9579	24.1371	23.7783	23.8927	22.0775	22.5775
6	23.9573	24.1364	23.7776	23.8924	22.0774	22.5778
7	23.9576	24.1359	23.7778	23.8916	22.0772	22.5770
8	23.9581	24.1359	23.7783	23.8914	22.0773	22.5766
9	23.9581	24.1373	23.7784	23.8911	22.0774	22.5749
10	23.9576	24.1363	23.7779	23.8912	22.0773	22.5755
11	23.9577	24.1356	23.7780	23.8918	22.0772	22.5752
12	23.9577	24.1339	23.7779	23.8922	22.0772	22.5758

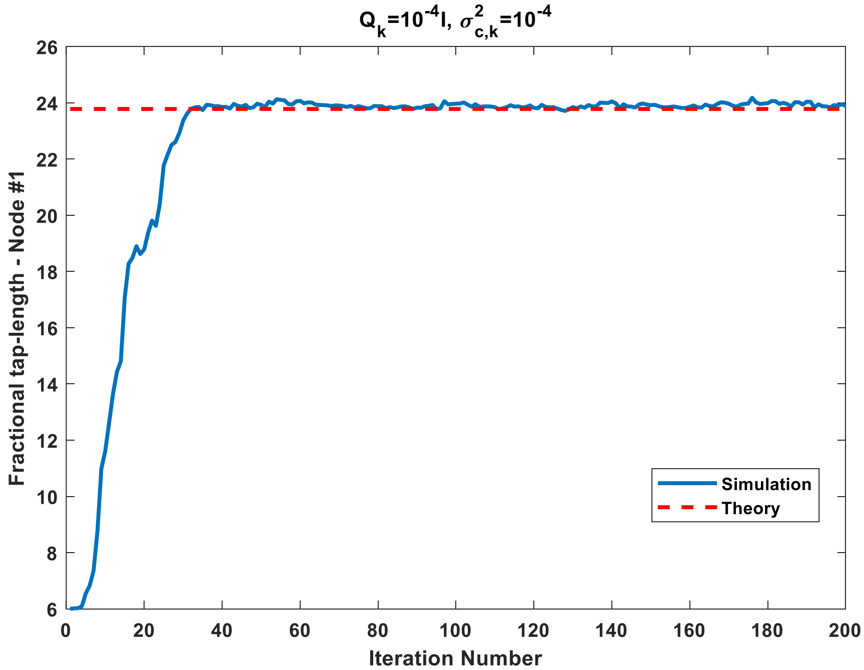

Fig. 2. Evolution curve of the fractional tap-length with the derived theoretical steady-state tap-length for node $k = 1$, and for $L_{opt} = 20$.

VI. CONCLUSIONS

This paper provided a steady-state analysis for the distributed incremental FT variable tap-length

LMS algorithm under the noisy link conditions. Based on this analysis, we derived a mathematical formulation for the steady-state tap-length at each node. Numerical experiments confirmed that there is a good match between simulation results and our theoretical derived expressions. Our derived equations show how the steady-state tap-length value is affected by noisy links. However, several critical results were induced. Firstly, as the noise level increases, the steady-state tap-length decreases compared to the ideal link version. However, under low noise conditions, this length is still larger than the optimal filter length. Secondly, there isn't any trace of the noise $\ell_{q,k,i}$ in the steady-state tap-length relation. The noise term added to the local estimation of the unknown vector, $\boldsymbol{q}_{k,i}$, only appears in this relation. So, $\ell_{q,k,i}$ doesn't have any effect on the steady-state tap-length. It could be explained by the fact that $\ell_{q,k,i}$ plays the same role as the leakage factor. In other words, proper selection of the leakage factor could compensate for the negative effect of $\ell_{q,k,i}$.

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