Polarization of Multi-Relay Channels: A Suitable Method for DF and CF Relaying with Orthogonal Receiver

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Abstract- Polar codes, that have been recently introduced by Arikan, are one of the first codes that achieved the capacity for vast numerous channels and they also have low complexity in symmetric memoryless channels. Polar codes are constructed based on a phenomenon called channel polarization. This paper discusses relay channel polarization in order to achieve the capacity and show that if inputs of two different relay channels follow the Arikan polarization structure, then they will be categorized as good and bad relay channels. Also, it has been shown that the eencoding and decoding complexity for these codes is $O(N \log N)$, N is the code-length , and their error probability is $O(2^{-(N)^{\beta}})$ like the Arikan's work, β is a number between [0,0.5]. In order to validate our construction of polar codes for relay channels, some numerical examples for this idea have been presented. Also, the efficiency of this construction for decode-and-forward and compress-and-forward relaying strategies have been analyzed by using simulation results for finite block length in relay channels with orthogonal receiver.

Index Terms- Polar code, Capacity, Channel polarization, Relay channel, Error probability, Complexity.

I. INTRODUCTION

The relay channel, introduced by Van der Meulen in [1], is a communication channel with a sender and receiver assistant in communication by using a relay node. A memoryless relay channel is defined by the probability distribution $W(Y_r, Y | X, X_r)$, where X is the symbol transmitted by the source, X_r is the symbol transmitted by the relay, Y_r is the symbol received by the relay and finally Y is the symbol received by the destination as shown in Fig. 1.

The message M has been assumed to be uniformly distributed over the message set and the average probability of error is defined as $P_e^{(N)} = \Pr\{\hat{M} \neq M\}$, N is the code-length. The rate R for this

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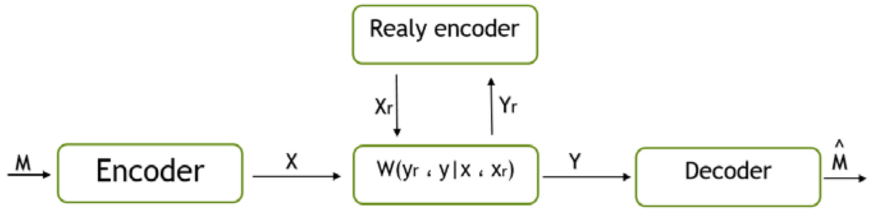

Figure 1. A relay channel with probability distribution $W(Y_r, Y|X, X_r)$

channel is achievable if there is a sequence of $(2^{NR}, N)$ codes such that the error probability of infinite block length converges to 0. Generally, the capacity of the relay channel is still an open problem.

Cover and El Gamal established an outer bound on the capacity which is known as the cut-set bound in [2]. The cut-set bound is given by:

$$C \leq \max_{p(x,x_r)} \min\{I(X, X_r; Y_r), I(X; Y, Y_r|X_r)\}. \tag{1}$$

Most of the important coding strategies for relay channels are based on two different philosophies of information processing at the relay: Decode-and-Forward (DF) and Compress-and-Forward (CF). For DF strategy, the relay recovers the message transmitted by the source and forwards some information about it to the destination that complements the observation obtained through the source-destination link. The DF lower bound is given by:

$$C \geq R_{DF} := \max_{p(x,x_r)} \min\{I(X, X_r; Y_r), I(X; Y_r|X_r)\}, \tag{2}$$

The above equation, (2), can be rewritten in another way. The DF rate can be expressed as follows:

$$R_{DF} = \max_{p(v)p(x_r),p(v,x_r)} \min\{I(X_r;Y) + I(V;YX_r), I(V;Y_rX_r)\}, \tag{3}$$

that we define V~Unif [1:q] for some prime q. For CF strategy, the relay helps to communicate by sending a description of its received sequence to the receiver. This description is correlated with the received sequence; therefore, Wyner-Ziv coding is used to reduce the rate that is needed to communicate the received sequence to the receiver. The CF lower bound is given by:

$$C \geq R_{CF} := \max I(X; \hat{Y}_r, Y, X_r), \tag{4}$$

where the maximum is taken over pmfs condition $p(x)p(x_r)p(\hat{y}_r|x_r, y_r)$ such that $I(X_r;Y) \geq I(Y_r;\hat{Y}_r|X_r,Y)$, where $\hat{Y}_r$ is a estimation of the symbol received by the relay.

Recently, polar codes have been introduced by Arikan through a phenomenon called channel polarization and these type of codes have been shown to be one of the first kind of codes that achieve the capacity for binary input symmetric channels [3]-[4]. Polar codes and polarization scheme have been extended to various multi-terminal scenarios, such as the Multiple-Access Channels [5-7], Broadcast Channels [8,9], etc.

This paper presents polar coding scheme and channel polarization phenomenon for relay channel. In addition, it shows how the polar coding for the relay channel can be proposed and will explain the relay channel polarization by using the polarization of cut-set bound. The proposed schemes possess the standard properties of polar codes with regards to encoding and decoding, which can be performed with complexity $O(N.\log N)$. Also, for this scheme the block error probability is an exponential function of the block length, likewise $O(2^{-(N)^{\beta}})$ for any $0 \leq \beta \leq 0.5$. The second part of this paper presents some numerical examples for this constructions in order to validate our idea for analyzing the polar codes for relay channels. Also, the efficiency of using polar codes for DF and CF relaying by using simulation results in relay channels with orthogonal receiver has been analysed.

This paper is organized as follows: In Section II, the background on polar codes and previous works on relay channels with respect to polar coding are reviewed. It also contains the achieved result of the DF and CF strategy in general three-node relay channel using polar codes shown in this section. In Section III, polarization of relay channel for cut-set bound is proven. In Section IV, polar codes were shown to be proper for polarized relay channels with orthogonal receiver and finally, Section V concludes the paper.

II. POLAR CODES AND RELAY CHANNEL

In this section of the paper, a brief overview of the groundbreaking work of Arikan [3] on polar codes and channel polarization is studied and a brief part of previous works on relay channels with respect to polar coding are presented.

A. Polar codes

Polar codes are constructed based upon a phenomenon called polarization [4]. The basic polarization matrix is given as $G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. The Kronecker power of G is defined for any n f 1 according to $G_2^{\otimes n} = \begin{bmatrix} G_2^{\otimes(n-1)} & 0 \\ G_2^{\otimes(n-1)} & G_2^{\otimes(n-1)} \end{bmatrix}$. For a DMC with a binary-input output-symmetric can be defined as a channel splitting map $(W, W) \rightarrow (W^-, W^+)$ by following [3]. The synthesized channels $W^-: F_2 \rightarrow Y^2$ and $W^+: F_2 \rightarrow F_2 \times Y^2$ which F_2 is $\{0,1\}$, are given by:

$$W^{-}(y_1^2 | u_1) = \sum_{u_2 \in \{0,1\}} \frac{1}{2} W(y_1 | u_1 \oplus u_2) W(y_2 | u_2), \tag{5}$$

$$W^{+}(y_1^2, u_1 | u_2) = \frac{1}{2} W(y_1 | u_1 \oplus u_2) W(y_2 | u_2). \tag{6}$$

After the channel splitting to: W^{-} and W^{+}, which are noisier and more reliable as compared to the original channel W , respectively. In order to measure how good a binary-input channel is W , Arikan uses the Bhattacharyya parameter for a channel W denoted by $Z(W)$ and it has been defined as follows [3]:

$$Z(W) = \sum_{y \in Y} \sqrt{W(y|0) W(y|1)}. \tag{7}$$

Channels with $Z(W)$ close to zero are almost noiseless while channels with $Z(W)$ close to one are almost pure-noisy channels [4]. Let $[N]$ denote the set of positive integers less than or equal to N . The set of good bit-channels $I_N(W)$ is defined as follows for any $0 \le \beta \le 0.5$:

$$I_N(W) := \{i \in [N] : Z(W_{2N}^{(i)}) \le \frac{2^{-(N)^{\beta}}}{N}\}. \tag{8}$$

Then, the channel polarization theorem is exhibited by showing that the fraction of good bit-channels, approaches the symmetric capacity $I(W)$ as N goes to infinity [3]-[4]. $I(W)$ is the mutual information between the input and output of W when input distribution has been considered to be uniform. Eq. (8) leads to the construction of capacity-achieving polar codes [3]. For any $0 \le \beta \le 0.5$, the reliability of polar codes is determined by error-probability block under the Successive Cancellation (SC) decoding yields:

$$P_e = \sum_{i \in A} Z(W_N^{(i)}) = o(2^{-2^{n\beta}}). \tag{9}$$

Where the set $A = \{i \in [N] : Z(W_N^{i}) \in [0, \delta]\}$ is characterized as follows, where δ f 0.

B. Previous works

In polar codes introduced by Arikan, two methodologies were presented. The main idea was that the capacity achieving codes for a large case of channels can be constructed and also $P_N^{(i)}$ approaches the error-free channel or a completely noisy channel when N grows. The rate of error-free channels approaches the channel capacity. The main purpose of the current research that proposed the polar codes with relay channel is to design the capacity achieving codes. The first application of polar codes for the relay chennels was reported in [9]-[10]. In this study, it has been shown that polar codes can achieve the capacity of symmetric physically degraded relay channels with binary input. Also, [11] developed polar code in order to achieve the capacity of DF relaying in binary symmetric relay channel, which is stochastically degraded with orthogonal receivers. It has been shown that polar codes can achieve the capacity of binary input symmetric degraded relay channel in studies by [12]-[13]. Also, the achievability of the two other lower bounds using polar coding techniques was shown in [13]. A new scheme in order to choose proper indices for sending the information bits in a polarized relay channel has been presented in [14].

In all the dicussed papers, capacity achieving is the main purpose for using polar codes. The capacity increases regarding to $N \to \infty$ just in [5]-[6]-[7]. This approach shows that the capacity of one of the polarized channels increased while the other decreased with respect to the main channel. In [7], a novel method for polarization a MAC was demonstrated and it has been shown that polarization of a general MAC with a point-to-point channel can give a larger achievable rate region. In contrast to the first approach, the main idea behind the second approach was that by increasing the number of channels, N , the capacity region would increase and it is worth mentioning as an Information Theoretic point of view. In this paper, by using the second methodology via two relays, the capacity of one relay increases while the other decreases and the capacity region of the relay changes when the links are polarized. This idea is demonstrated in three scenarios in section III.

C. DF and CF Relaying using Polar Codes

This section discusses recent researches about DF and CF relaying using polar codes briefly. Polar coding schemes for DF strategy have been proposed in degraded relay channels, where $X \to (X_R, Y_R) \to Y$ construct a Markov chain, [10]-[13]. Under the degradation condition, the two polarization processes involved in the DF scheme possess a nested structure. A polar coding scheme for CF has been proposed in relay channels with orthogonal receiver components, where $Y = (Y', Y'')$ and $W(y_r, y \mid x, x_r) = W(y', y_r \mid x) W(y'' \mid x_r)$ in [11]. This scheme achieves the so-called symmetric CF rate. The main result for this section is that for any transmission rate $R \prec R_{DF}$ and any fixed rate $R \prec R_{CF}$, there exists a sequence of polar codes with block error probability $P_e^{(n)} = \Pr\{\hat{M} \neq M\}$ under SC decoding bounded as $P_e \leq O(2^{-(N)^{\beta}})$ for any $0 \leq \beta \leq 0.5$ in a stochastically degraded relay channel with DF strategy and for a relay channel with CF strategy with orthogonal receiver components, respectively in [11]-[14].

III. POLARIZATION FOR RELAY CHANNEL

The main idea of channel polarization is extended to relay channel in this section, in which this technique is described in order to polarize a given binary-input relay channel same as studied by [3], [5-8]. The polarization of cut-set bound is also proven. It is shown that after polarization of two relay channels, the capacity of one relay increases while the other decreases and the capacity region of the relay changes for cut-set rate.

Theorem 1. (Rate polarization for two relay channels) Considering Fig.2, there are three scenarios to represent rate bounds of R^- and R^+ for two relays channel after polarization as follows:

$$R^- = \min\begin{cases} \{I(U_1V_1;Y_1^2), I(U_1;Y_{r_1}^2Y_1^2 \mid V_1^2)\} & ,(S_1,S_2)=(1,1) \\ \{I(U_1V_1;Y_1^2), I(U_1;Y_{r_1}^2Y_1^2 \mid V_1^2)\} & ,(S_1,S_2)=(1,0) \\ \{I(U_2V_2;Y_1^2V_1), I(U_2;Y_{r_1}^2Y_1^2 \mid V_2)\} & ,(S_1,S_2)=(0,1) \end{cases} \tag{10}$$

and

$$R^+ = \min\begin{cases} \{I(U_2V_2;Y_1^2U_1V_1), I(U_2;Y_{r_1}^2Y_1^2U_1 \mid V_1^2)\} & ,(S_1,S_2)=(1,1) \\ \{I(U_1V_1;Y_1^2), I(U_1;Y_{r_1}^2Y_1^2 \mid V_1^2)\} & ,(S_1,S_2)=(1,0) \\ \{I(U_2V_2;Y_1^2V_1), I(U_2;Y_{r_1}^2Y_1^2 \mid V_2)\} & ,(S_1,S_2)=(0,1) \end{cases} \tag{11}$$

Binary parameter S is defined such that $S=1$ and $S=0$ are representatives of applying and not applying matrix G_2 for encoding section of input bits of source and relay, respectively. S_1 represents the case of input bits from the source, while S_2 represents the case of input bits from the relay.

Proof. Let us define, two independent uses of the channel W given in a relay channel W^2 as shown in Fig. 2. The cut-set bound in a channel W^2, is described by the two following quantities:

$$I(X_1X_2, X_{r_1}X_{r_2}; Y_1Y_2) = 2I_1(W) \tag{12}$$

and

$$I(X_1X_2; Y_{r_1}Y_{r_2}, Y_1Y_2 \mid X_{r_1}X_{r_2}) = 2I_2(W). \tag{13}$$

Also, the cut-set bound is as follows:

$$R \leq \min\{I(X_1^2, X_{r_1}^2; Y_1^2), I(X_1^2; Y_1^2, Y_{r_1}^2 \mid X_{r_1}^2)\}., \tag{14}$$

As shown in Fig. 2., we have; $X_1^2 = U_1^2G_2$ and $X_{r_1}^2 = V_1^2G_2$. Now we get:

$$\begin{aligned} 2I_1(W) &= I(X_1X_{r_1}, X_2X_{r_2}; Y_1Y_2) = I(U_1U_2V_1V_2; Y_1Y_2) = I(U_1V_1; Y_1Y_2) + I(U_2V_2; Y_1Y_2 \mid U_1V_1) = \\ &I(U_1V_1; Y_1Y_2) + I(U_2V_2; Y_1Y_2, U_1V_1) = I_1(W^-) + I_1(W^+) \end{aligned} \tag{15}$$

and

$$\begin{aligned} 2I_2(W) &= I(X_1X_2; Y_{r_1}Y_{r_2}; Y_1Y_2 \mid X_{r_1}X_{r_2}) = I(U_1U_2; Y_{r_1}Y_{r_2}; Y_1Y_2 \mid V_1V_2) = I(U_1; Y_{r_1}Y_{r_2}; Y_1Y_2 \mid V_1V_2) \\ &+ I(U_2; Y_{r_1}Y_{r_2}; Y_1Y_2, U_1 \mid V_1V_2) = I_2(W^-) + I_2(W^+). \end{aligned} \tag{16}$$

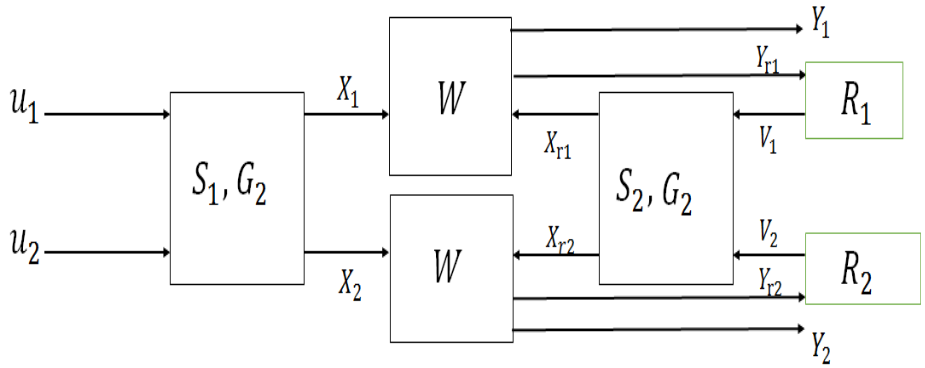

Fig. 2. Relay channel W^2 formed by combining two relay channel.

As shown, $I_1(W^-)$ and $I_2(W^-)$ indicate proper bounds for relay channel $U_1 \times V_1 \to Y_1 Y_2 Y_{r_1} Y_{r_2}$; and $I_1(W^+)$ and $I_2(W^+)$ also indicate proper bounds for relay channel $U_2 \times V_2 \to Y_1 Y_2 Y_{r_1} Y_{r_2} U_1 | V_1 V_2$. Let channel $W : X \times X_r \to Y \times Y_r$ be a relay channel with $\{0,1\}$ input alphabet. We define two new relay channels as $W^- : X \times X_r \to Y^2 \times X_r^2$ and $W^+ : X \times X_r \to Y^2 \times Y_r^2 \times X \times X_r$, in which we have:

$$W^-(y_1^2, y_{r_1}^2 | u_1, v_1) = \sum_{u_2 \in X_2, v_2 \in X_r} \frac{1}{4} W(y_1, y_{r_1} | u_1 \oplus u_2 y_1 \oplus v_2) W(y_2, y_{r_2} | u_2 v_2) \tag{17}$$

and

$$W^+(y_1^2, y_{r_1}^2, u_1, v_1 | u_2, v_2) = \frac{1}{4} W(y_1, y_{r_1} | u_1 \oplus u_2, v_1 \oplus v_2) W(y_2, y_{r_2} | u_2 v_2) \tag{18}$$

where W^- and W^+ are correspond to relay channels $W^- : U_1 \times V_1 \to Y_1 Y_2 Y_{r_1} Y_{r_2}$ and $W^+ : U_2 \times V_2 \to Y_1 Y_2 Y_{r_1} Y_{r_2} U_1 | V_1 V_2$, respectively. Note that W^- is a bad channel and W^+ is a good channel as compared to an original channel W. Also for capacity bound, we get:

$$I_i(W^-) \le I_i(W) \le I_i(W^+); i = 1,2 \tag{19}$$

To prove them, since $I_1(W^+) \ge I_1(W)$ and $I_2(W^+) \ge I_2(W)$, then, according to (11) and (12), it should be $I_i(W^-) \le I_i(W) \le I_i(W^+); i = 1,2$. Correspondingly, for R, we get:

$$R(W^-) \text{ p } R(W) \text{ p } R(W^+) \tag{20}$$

Now, we define parameter S such that $S = 1$ and $S = 0$ are representatives of to apply and not to apply matrix G_2 for encoding section of input bits from the source and relay, respectively. S_1

represents the case relevant to the input bits from the source, while S_2 represents the case relevant to the input bits from the relay.

$$(S_1,S_2)=\begin{cases}(1,1) & ,i=1\\ (1,0) & ,i=2\\ (0.1) & ,i=3\end{cases} \tag{21}$$

However, in this article, they are only investigated for cut-set bound:

$$R_i^- = \min\begin{cases}\{I(U_1V_1;Y_1^2), I(U_1;Y_{r_1}^2Y_1^2|V_1^2)\} & ,i=1\\ \{I(U_1V_1;Y_1^2), I(U_1;Y_{r_1}^2Y_1^2|V_1^2)\} & ,i=2\\ \{I(U_2V_2;Y_1^2V_1), I(U_2;Y_{r_1}^2Y_1^2|V_2)\} & ,i=3\end{cases} \tag{22}$$

and

$$R_i^+ = \min\begin{cases}\{I(U_2V_2;Y_1^2U_1V_1), I(U_2;Y_{r_1}^2Y_1^2U_1|V_1^2)\} & ,i=1\\ \{I(U_1V_1;Y_1^2), I(U_1;Y_{r_1}^2Y_1^2|V_1^2)\} & ,i=2\\ \{I(U_2V_2;Y_1^2V_1), I(U_2;Y_{r_1}^2Y_1^2|V_2)\} & ,i=3\end{cases} \tag{23}$$

Lemma 1. For analyzing (10-11) and (22-23), we have the following inequalities, Firstly, for

$$I(U_2V_2;Y_2)\in[0,\varepsilon)\cup(1-\varepsilon,1], \tag{24}$$

we have:

$$I(U_2V_2;Y_1Y_2U_1V_1)-I(U_2V_2;Y_2)\prec\delta, \tag{25}$$

also for

$$I(U_2;Y_{r_2}Y_2|V_2)\in[0,\varepsilon)\cup(1-\varepsilon,1] \tag{26}$$

we have

$$I(U_2;Y_{r_1}Y_{r_2}Y_1Y_2U_1|V_1V_2)-I(U_2;Y_{r_2}Y_2|V_2)\prec\delta. \tag{27}$$

Proof. For a channel with input binary and arbitrary output alphabet β we have $W:\{0,1\}\to\beta$ and $I(A_2;B_1B_2A_1)-I(A_2;B_2)\prec\delta$, then $I(A_2;B_2)\in[0,\varepsilon)\cup(1-\varepsilon,1]$, that A_1,A_2,B_1,B_2 are random variables jointly distributed as

$$W_{A_1,A_2,B_1,B_2}(a_1,a_2,b_1,b_2)=\frac{1}{4}W(b_1|a_1\oplus a_2)W(b_2|a_2); \tag{28}$$

then there is a $\delta:=\delta(\varepsilon)\succ 0$, where $\varepsilon\succ 0$ [7,17]. Note that this can be chosen irrespective of the alphabet β. Therefore, it is concluded that for any $\varepsilon\succ 0$, there is $\delta\succ 0$ such that if W is a binary input relay channel with $I_1(W^+) - I_1(W) < \delta$, then $I_1(W) \in [0,\varepsilon) \cup (1-\varepsilon,1]$. Similarly, if W is such that $I_2(W^+) - I_2(W) < \delta$, then $I_2(W) \in [0,\varepsilon) \cup (1-\varepsilon,1]$. For $i=1$, with considering $A_i = U_i V_i$ and $B_i = Y_i$, we can conclude that

$$I(U_2V_2;Y_2) \in [0,\varepsilon) \cup (1-\varepsilon,1] \tag{29}$$

and

$$I(U_2V_2;Y_1Y_2U_1V_1) - I(U_2V_2;Y_2) < \delta; \tag{30}$$

and with define $A_i = U_i$ and $B_i = Y_{r_i}Y_i|V_i$ can be concluded that

$$I(U_2;Y_{r_2}Y_2|V_2) \in [0,\varepsilon) \cup (1-\varepsilon,1] \tag{31}$$

and

$$I(U_2;Y_{r_1}Y_{r_2}Y_1Y_2U_1|V_1V_2) - I(U_2;Y_{r_2}Y_2|V_2) < \delta. \tag{32}$$

Lemma 2. For analyzing (10-11) and (22-23), we have,

$$I(U_1V_1;Y_1) \in [0,\varepsilon) \cup (1-\varepsilon,1] \tag{33}$$

and

$$I(U_1V_1;Y_1Y_2) - I(U_1V_1;Y_1) < \delta; \tag{34}$$

also

$$I(U_1;Y_{r_1}Y_1|V_1) \in [0,\varepsilon) \cup (1-\varepsilon,1] \tag{35}$$

and

$$I(U_1;Y_{r_1}Y_{r_2}Y_1Y_2|V_1V_2) - I(U_1;Y_{r_1}Y_1|V_1) < \delta. \tag{36}$$

Proof. The proof of this lemma is similar to the proof of lemma 1.

Lemma 3. For analyzing (10-11) and (22-23), we have,

$$I(U_2V_2;Y_2) \in [0,\varepsilon) \cup (1-\varepsilon,1] \tag{37}$$

and

$$I(U_2V_2;Y_1Y_2V_1) - I(U_2V_2;Y_2) < \delta; \tag{38}$$

also

$$I(U_2;Y_{r_2}Y_2|V_2) \in [0,\varepsilon) \cup (1-\varepsilon,1] \tag{39}$$

and

$$I(U_2;Y_{r_1}Y_{r_2}Y_1Y_2|V_2) - I(U_2;Y_{r_2}Y_2|V_2) \prec \delta. \quad (40)$$

Proof. The proof of this lemma is similar to the proof of lemma 1.

Now suppose W is a binary input relay channel, let $\{B_n\}_{n\geq 1}$ be an *i.i.d.* uniform random variable valued in {-,+}, with $\Pr(B_1 = -) = \Pr(B_1 = +) = \frac{1}{2}$; we defined a relay channel valued random process $\{W_n : n \geq 0\}$ via,

$$W_0 := W, \quad W_n := W_{n-1}^{B_n}, \quad n \geq 1 \quad (41)$$

Further, we defined random processes $\{I_{1_n} : n \geq 0\}$ and $\{I_{2_n} : n \geq 0\}$ such that:

$$I_{1_n} := I_1(W_n), \quad I_{2_n} := I_2(W_n). \quad (42)$$

Since W_n is a binary input relay channel, $I_1(W_n)$ and $I_2(W_n)$ take values in [0,1]; hence the mentioned processes are bounded. The martingale claims follow from (15) and (16), respectively. The process $(I_1(W_n), I_2(W_n))$ converges almost surely, and the limit $(I_{1\infty}, I_{2\infty}) = \lim_{n\to\infty}(I_1(W_n), I_2(W_n))$. Therefore, The processes $\{I_1(W_n) : n \geq 0\}$ and $\{I_2(W_n) : n \geq 0\}$ are the bounded martingale.

The following theorem is presented for Bhattacharya parameter of relay channels polarized.

Theorem 2. Consider each relay channel with $I_1(W) = I_{MAC}$ and $I_2(W) = I_{BC}$, for the *BC phase*, we have:

$$W_{BC} \rightarrow \begin{cases} W_{BC}^{-} & , Z^{-}(W) \leq 2Z_{BC}(W) \\ W_{BC}^{+} & Z^{+}(W) \leq Z_{BC}^{2}(W) \end{cases} \quad (43)$$

also, for MAC phase of the relay channel, we have:

$$W_{MAC} \rightarrow \begin{cases} W_{MAC}^{-} & , Z^{-}(W) \leq 2Z_{MAC}(W) \\ W_{MAC}^{+} & Z^{+}(W) \leq Z_{MAC}^{2}(W) \end{cases} \quad (44)$$

Proof. The proof is similar to the proof of Theorem 3 in [7]. The fact is that that for any binary input discrete memoryless channel W, we have $I(W) + Z(W) \geq 1$, being used; in fact $I(W)^2 + Z(W)^2 \leq 1$ [4]. In polarizing mode, using [3] we get:

$$W^{-} : Z^{-}(W) \leq 2Z(W) - Z(W)^2, \quad (45)$$

$$W^{+} : Z^{+}(W) = Z(W)^2; \quad (46)$$

and by using [5]-[8] it can be written as $Z^{-}(W) \leq 2Z(W)$ and $Z^{+}(W) \leq Z(W)^2$. The relay channel is combined with two *MAC* and *BC* channels and also it can be considered as a point-to-point channel. Regarding [19], the following relations can written for *MAC* and *BC phase* of relay channel, respectively. For *BC phase*, we have:

$$W_{BC} \rightarrow \begin{cases} Z_1^-(W) \le 2Z_{BC}(W), & W_{BC}^- \\ Z_1^+(W) \le Z_{BC}(W)^2, & W_{BC}^+ \end{cases} \tag{47}$$

also, for MAC phase, we have:

$$W_{MAC} \rightarrow \begin{cases} Z_2^-(W) \le 2Z_{MAC}(W), & W_{MAC}^- \\ Z_2^+(W) \le Z_{MAC}(W)^2, & W_{MAC}^+ \end{cases} \tag{48}$$

So, in general, we have:

$$Z(W^-) \le \max\{Z_1(W^-), Z_2(W^-)\} \le \max\{2Z_{BC}(W), 2Z_{MAC}(W)\}, \tag{49}$$

and

$$Z(W^+) \le \max\{Z_1(W^+), Z_2(W^+)\} \le \max\{Z_{BC}(W)^2, Z_{MAC}(W)^2\}. \tag{50}$$

Example 1: Let us consider that all links in Fig. 2 are binary erasure channel. According to Theorem 1, one can find the capacity bound of polarized relay channels after polarization. For a binary erasure relay channel where the erasing probability between sender and relay, sender and destination, and relay and destination is ε_1, ε_2 and ε_3 respectively, the capacity region is bounded as:

$$R \le \max_{\beta} \min\{I(W_1), I(W_2)\} = \max_{\beta} \min\{(1-\varepsilon_1\varepsilon_2), (1-\varepsilon_2) + \beta(1-\varepsilon_3)\} \tag{51}$$

where β and ε_i are coupling and erasure parameter respectively, and they are between 0 and 1 [20]. Coupling parameter describes the proportion of information that has been sent via sender and it is available at the relay.

Now, for case $(S_1, S_2) = (1,1)$ in Theorem 1, one can find out that after polarization the capacity region is bounded by two quantities R^- and R^+ as follows:

$$R^- \le \max_{\beta} \min\left\{I(W_1^-), I(W_2^-)\right\} = \max_{\beta} \min\left\{1-\varepsilon_1\varepsilon_2(2-\varepsilon_1)(2-\varepsilon_2), (1-2\varepsilon_2+\varepsilon_2^2) + \beta(1-2\varepsilon_3+\varepsilon_3^2)\right\} \tag{52}$$

and

$$R^+ \le \max_{\beta} \min\left\{I(W_1^+), I(W_2^+)\right\} = \max_{\beta} \min\left\{1-(\varepsilon_1\varepsilon_2)^2, (1-\varepsilon_2^2) + \beta(1-\varepsilon_3^2)\right\}. \tag{53}$$

The aforementioned relations were evaluted using Theorem 1 and equalities (35) and (36). In these relations, clearly, we have:

$$1-(\varepsilon_1\varepsilon_2)(2-\varepsilon_1)(2-\varepsilon_2) \le 1-\varepsilon_1\varepsilon_2 \le 1-(\varepsilon_1\varepsilon_2)^2 \tag{54}$$

and

$$(1-(2\varepsilon_2-\varepsilon_2^2)) + \beta(1-(2\varepsilon_3-\varepsilon_3^2)) \le (1-\varepsilon_2) + \beta(1-\varepsilon_3) \le (1-\varepsilon_2^2) + \beta(1-\varepsilon_3^2) \tag{55}$$

Table1. The value of capacity for different channel parameter

$I(W^-_{mac})$	$I(W^-_{bc})$	$I(W^+_{mac})$	$I(W^+_{bc})$	$I(W_{mac})$	$I(W_{bc})$
0.5	0.4375	1.5	0.9375	1	0.75

Then, it can be concluded that $I(W_1^-) \le I(W_1) \le I(W_1^+)$ and $I(W_2^-) \le I(W_2) \le I(W_2^+)$. Therfore, after polarization, we have: $R(W^-)\ \mathrm{p}\ R(W)\ \mathrm{p}\ R(W^+)$.

Example 2: Consider the relay channel, as shown in Fig. 2. Let $W_{SR} = W_{SD} = W_{RD} = W$, where $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon = BEC(0.5)$, for binary erasure relay channel from the basic polarization phenomenon and by using Theorem (3.2) in [19] and the relation that have been extracted from Example 1, we will have the following equalities:

$$I(W_{bc}^-) = 1-(\varepsilon_1\varepsilon_2)(2-\varepsilon_1)(2-\varepsilon_2) \xrightarrow{\varepsilon_1=\varepsilon_2=\varepsilon} = 1-(\varepsilon(2-\varepsilon))^2 \xrightarrow{\varepsilon=0.5} I(W_{bc}^-) = 0.4375, \tag{56}$$

$$I(W_{mac}^-) = (1-(2\varepsilon_2-\varepsilon_2^2)) + \beta(1-(2\varepsilon_3-\varepsilon_3^2)) \xrightarrow[\beta=1]{\varepsilon_2=\varepsilon_3=\varepsilon} = 2(1-2\varepsilon+\varepsilon^2) \xrightarrow{\varepsilon=0.5} I(W_{mac}^-) = 0.5, \tag{57}$$

and

$$I(W_{bc}^+) = 1-(\varepsilon_1\varepsilon_2)^2 \xrightarrow{\varepsilon_1=\varepsilon_2=\varepsilon} = 1-\varepsilon^4 \xrightarrow{\varepsilon=0.5} I(W_{bc}^+) = 0.9375, \tag{58}$$

$$I(W_{mac}^+) = (1-\varepsilon_2^2) + \beta(1-\varepsilon_3^2) \xrightarrow[\beta=1]{\varepsilon_2=\varepsilon_3=\varepsilon} = 2(1-\varepsilon^2) \xrightarrow{\varepsilon=0.5} I(W_{mac}^+) = 1.5. \tag{59}$$

Also, always for any $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$, we have following relations:

$$I(W_{MAC}^-) + I(W_{MAC}^+) = (1-2\varepsilon+\varepsilon^2) + \beta(1-2\varepsilon+\varepsilon^2) + (1-\varepsilon^2) + \beta(1-\varepsilon^2) = 2(1-\varepsilon)(1+\beta) \le 2I(W_{MAC}) \tag{60}$$

and

$$I(W_{BC}^-) + I(W_{BC}^+) = (1-(\varepsilon(2-\varepsilon))^2) + (1-\varepsilon^4) = 2-2\varepsilon^2(\varepsilon^2-2\varepsilon+2) \le 2-2\varepsilon^2 = 2I(W_{BC}). \tag{61}$$

So, it can be concluded that, values are reported in Table1,

$$I(W_{MAC}^-) + I(W_{MAC}^+) \le 2I(W_{MAC}) \tag{62}$$

and

$$I(W_{BC}^-) + I(W_{BC}^+) \le 2I(W_{BC}) \tag{63}$$

for any relay channel that each links are polarized; and

$$R \le \min\{I(W_{BC}), I(W_{MAC})\} = 1 - \max\{Z(W_{BC}), Z(W_{MAC})\}. \tag{64}$$

Lemma 4. Let's the channel between source and receiver be defined as W_{SD}, and the channel between source and relay be defined as W_{SR}. Then, A_{SD} and A_{SR} are information sets that resulted from polar codes of W_{SD} and W_{SR} channels, respectively. For any two discrete W_{SD} and W_{SR} channels memoryless, if W_{SD} is degraded of W_{SR}, then $A_{SD} \subseteq A_{SR}$.

Proof. The proof of this Lemma is like the proofs of Theorem 2 by [10] and Theorem 1 by [13]. According to Lemma 4 and polar codes construction, let us define m_0 to be the determined information vector at a source, and $m_0 = u_{A_{SR}}$, where u is the length of N from the input source; and $u_{A_{SR}}$ is the determined sub-vector $[u_{i \in A_{SR}}]$. In addition, we define m_{01} as $m_{01} = u_{A_{SR} \backslash A_{SD}}$, in where $u_{A_{SR} \backslash A_{SD}}$ is the sub-vector $[u_{i \in A_{SR} \backslash A_{SD}}]$ and $A_{SR} \backslash A_{SD}$ displays information set for W_{SR} and frozen bits set for W_{SD}; also, we define $m_{02} = u_{A_{SD}}$, where $u_{A_{SD}}$ is the sub-vector $[u_{i \in A_{SD}}]$ and A_{SD} means information set for W_{SD} and W_{SR} channels. In this schema, if no transition error occurs in W_{SR}, then $m_1 = m_{01}$. Now, let m_2 be a new message which is sent from source for *MAC phase* of the relay channel. $\hat{m}_1$ and $\hat{m}_2$ indicate estimations of m_1 and m_2 messages, respectively. Theorem 3 is represented with attention to lemma 2 and a special case that explains it.

Theorem 3. For any $\beta \le 0.5$ and sufficiently large block length of N, for case $r = 0$ and $r = 1$, the upper bound of the error probability under *SC* decoding, will be:

$$P_e \le O(2^{-(N)^{\beta}}). \tag{65}$$

That is r indicates the probability of the correlation between the information transition and the source, the complexity for this code is $O(N.\log N)$.

Proof. Let parameter E provides the error probability of $\{(\hat{m}_1, \hat{m}_2) \ne (m_1, m_2)\}$ for case $r = 0$, or $\{(\hat{m}_0 \ne m_0)\}$ for case $r = 1$; then we get:

$$P_e = \Pr(E_{BC})\Pr(E \,|E_{BC}) + \Pr(E_{BC}^{C})\Pr(E \,|E_{BC}^{C}) \tag{66}$$

where E_{BC} is $\{(\hat{m}_0 \ne m_0)\}$ on the relay, and E_{BC}^{C} is its complementary event; then we get:

$$\begin{aligned}
&= \Pr(E_{BC})\Pr(E \,|E_{BC}) + \Pr(E_{BC}^{C})\Pr(E \,|E_{BC}^{C}) \le \Pr(E_{BC}) + \Pr(E \,|E_{BC}^{C}) \\
&\le \Pr(E_{BC}) + \Pr(E_{MAC} \,|E_{BC}^{C})\Pr(E \,|E_{BC}^{C}, E_{MAC}) + \Pr(E_{MAC}^{C} \,|E_{BC}^{C})\Pr(E \,|E_{BC}^{C}, E_{MAC}^{C}) \\
&\le \Pr(E_{BC}) + \Pr(E_{MAC} \,|E_{BC}^{C}) + \Pr(E \,|E_{BC}^{C}, E_{MAC}^{C})
\end{aligned} \tag{67}$$

E_{MAC} represents an event $\{(\hat{m}_1, \hat{m}_2) \neq (m_1, m_2)\}$ for $r = 0$, or $\{\hat{m}_1 \neq m_1\}$ for $r = 1$; also E_{MAC}^C represents its complementary event. From (9) and [8], it is well-known that $P(E_{BC}) \leq O(2^{-(N)^\beta})$. For the rest of the terms in the right side of (67), $\Pr(E_{MAC} \mid E_{BC}^C)$ represents an event that $\{(\hat{m}_1, \hat{m}_2) \neq (m_1, m_2)\}$ on the receiver when $r = 0$, if there is no decoding error on a relay at the end of the *BC phase*. When $r = 1$, $\Pr(E_{MAC} \mid E_{BC}^C)$ represents an event $\{\hat{m}_1 \neq m_1\}$ on the receiver, if there is no decoding error on the relay. The upper bound of error probability for *MAC phase*, which is a transition under SC decoding, is $\Pr(E \mid E_{BC}^C, E_{MAC}^C)$, and upper bound of error probability for the last term of (67) is $\Pr(E \mid E_{BC}^C, E_{MAC}^C) \leq O(2^{-(N)^\beta})$. This part represents error probability for decoding message m_0, which has been sent to receiver in *BC phase*, and message m_1 is successfully decoded. From the aforementioned, it can be concluded that $\Pr(E_{MAC} \mid E_{BC}^C) + \Pr(E \mid E_{BC}^C, E_{MAC}^C) \leq O(2^{-(N)^\beta})$. Finally, regarding to (66), (67) and the aforementioned explanations, we can write:

$$P_e = \Pr(E) \leq O(2^{-(N)^\beta}). \tag{68}$$

The complexity $O(N.\log N)$ that is related to encoding and decoding can be achieved for the N relay channel.

IV. SIMULATION RESULTS

In this section, the proposed method has been analyzed and some simulation results are shown. First, the performance of polar codes for DF relaying in physically degraded relay channel, according to lemma 4, is shown in Fig.3. In this analysis, W_{SR} and W_{SD} are independent binary symmetric channels. Crossover probabilities for these links are equal to 0.05 and 0.15, respectively, and it has been considered as $I(W_{SR}) \approx 0.71$, $I(W_{RD}) \approx 0.53$ and $I(W_{SD}) \approx 0.31$. The BER performance for the case of a BSC as the relay-destination link is depicted. It has been observed that by increasing relay-destination link rate, the performance of DF relaying by using polar code is increasing while by increasing R_{RD} the performance is decreasing, because the relay-destination link dominates the errors. The destination cannot recover the additional information received by the relay that is necessary to decode the direct link observation; the effect of these errors becomes the bottleneck of the system.

Second, compres-and-forward relaying has been considered and the performance of polar codes for this case has been analysed. Crossover probabilities for source-relay and source-destination channels are independent binary symmetric channels. The BER for this case is as shown in Figure 4. In this case, it has been observed that increasing R_{RD} makes polar codes to perform better for CF relaying and decrease the BER for fixed n (block length) by increasing rate relay-destination link.

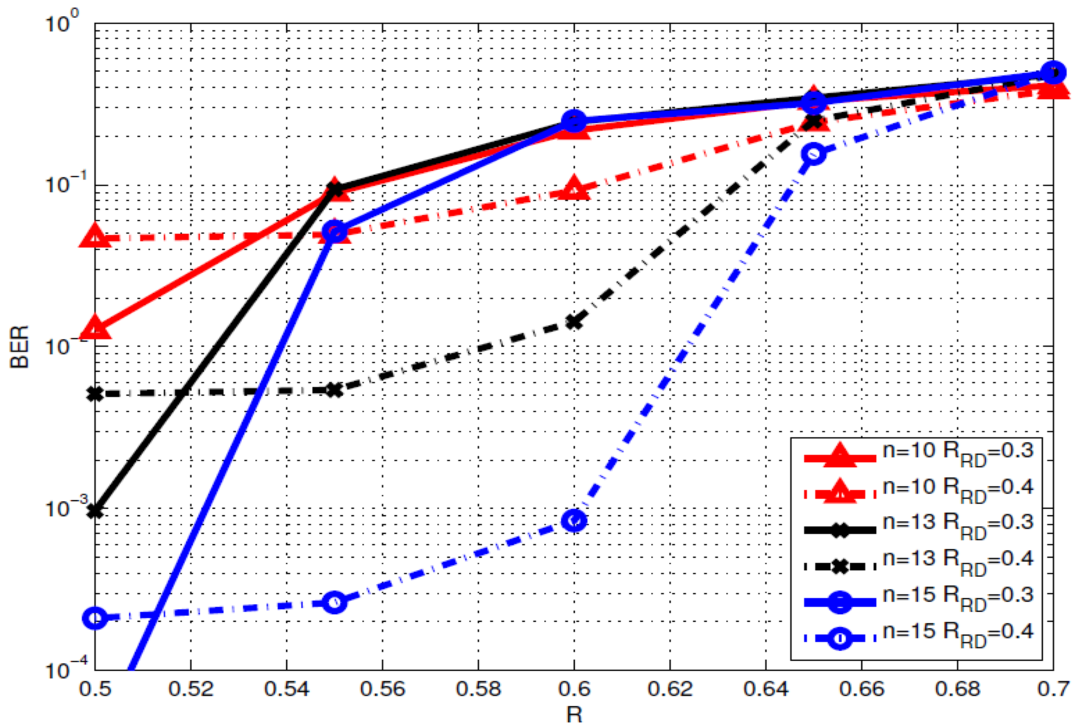

Fig. 3. Comparison of BER performance from polar codes for DF relaying for BSC with different R_{RD} .

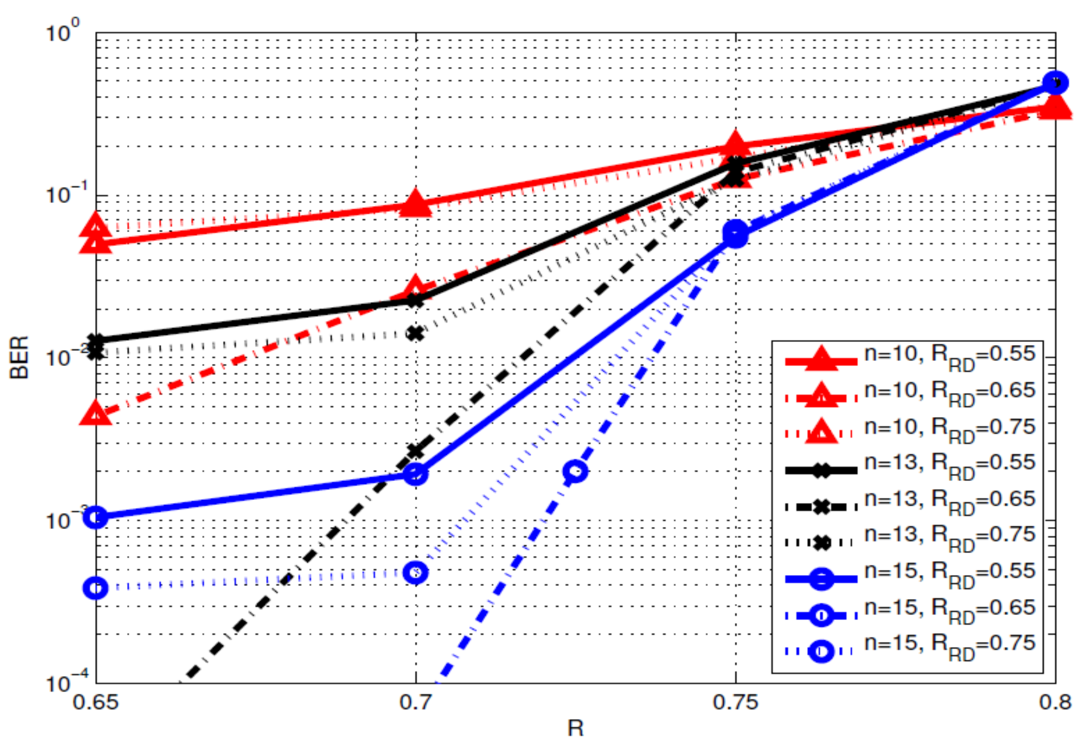

Fig. 4. Comparison of BER performance from polar codes for CF relaying for BSC with different R_{RD} .

Finally, the presented idea for polar codes in relaying channel is optimal for case $N \to \infty$. This behavior of polar codes in relay channels is the same to all constructions for finite block lengths of polar codes.

V. CONCLUSION

In this paper, polar codes have been shown to be suitable for DF and CF relaying with the orthogonal receiver via simulation results and represent channel polarization idea for relay channel case. They have been described for two relays, when the links are polarized, the capacity of one relay increases while the other decreases and the capacity region of the relay changes. Polarization has been shown to improve the cut-set bound for relay channel.

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